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**Abstract**

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**GEOPHYSICS**

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# THEORY OF SHOWER RAIN

The basic regularities of precipitation formation can be revealed with the aid of a cloud model with a constant ascending flux, first used by Ya. I. Frenkel and N. S. Shishkin ([1]). The study of droplet growth with allowance for gravitational coagulation and condensation, under the assumption of constant supersaturation in the cloud, was carried out by N. S. Shishkin ([2]). Howell' s numerical calculation ([3]) of the change of supersaturation with height without taking into account droplet coagulation indicates a very substantial dependence of supersaturation on height. In the present work the supersaturation is investigated with allowance for droplet coagulation.

In the absence of condensation, the change of supersaturation $c$ with height $z$, measured from the base of the cloud for $z \lesssim 2 \cdot 10^5$ cm, should be taken as

$$c(z) = Cz \qquad (C \simeq 2 \cdot 10^{-11} \text{ g/cm}^4), \tag{1}$$

since with height the temperature decreases and, consequently, so does the pressure of saturated vapor, which in accordance with the Clapeyron–Clausius equation changes faster than the atmospheric pressure described by the barometric formula.

In the presence of condensation, the change of supersaturation along the path $dz$ will be less than that calculated without allowing for condensation by the amount of moisture condensed during the time $dt = dz/w$ ($w$ is the velocity of the ascending air flow):

$$\frac{dc}{dz} = \rho \int_0^\infty n(v,z) \left(\frac{dv}{dt}\right)_{\text{cond}} \frac{dv}{w} = C, \tag{2}$$

where $n(v,z)$ is the number of droplets of volume $v$ at height $z = wt$. It is assumed here that the velocity of free fall of the droplets is much smaller than $w$. The rate of condensational growth is described by Maxwell' s formula:

$$\rho \left(\frac{dv}{dt}\right)_{\text{cond}} = 4\pi D c v^{1/3}, \tag{3}$$

where $\rho$ is the density of water, $D \simeq 0.1$ cm$^2$/sec is the effective diffusion coefficient, taking into account the slowing of condensation owing to the rise in droplet temperature.

Substituting formula (3) into equation (2), we obtain

$$\frac{dc}{dz} + \frac{4\pi D}{w}\, c \int_0^\infty n(v,z)\, v^{1/3} dv = C. \tag{4}$$

Since condensational growth of droplets is significant mainly at small heights, when the spectrum is close to monodisperse, one may replace

$$\int_0^\infty n(v,z)\, v^{1/3} dv \simeq N(z)\, [V(z)/N_0]^{1/3}, \tag{5}$$

where $N(z)$ is the total number of drops per unit volume at height $z$, $N_0 = N(0)$, and $V(z)$ is the water content, i.e., the water contained in drops at height $z$.

If the probability of collision of drops of volumes $u$ and $v$ is $b(u+v)$, where $b = \text{const}$,

$$N(z) = N_0 \exp\left\{-b/w \int_0^z V(z)\, dz\right\}. \tag{6}$$

But, since

$$\rho V(z) = Cz - c \tag{7}$$

in order to determine the water content, and with it the supersaturation, we have to find the solution of the following equation:

$$\rho \frac{dV}{dz} = \frac{4\pi D}{w} (Cz - \rho V)\, N_0^{2/3} V^{1/3} \exp\left\{-\frac{b}{w} \int_0^z V\, dz\right\}. \tag{8}$$

It is obvious that, as $z \to 0$,

$$V(z) \simeq \left(\frac{4\pi DC}{3w\rho}\right)^{3/2} N_0 z^3 \tag{9}$$

and, correspondingly, the supersaturation will be determined by formula (7).

In the absence of coagulation (i.e., for $b = 0$) it is easy to find the asymptotic behavior of the solution of equation (7) for large $z$:

$$\rho V(z) \simeq Cz - w\rho^{1/3}/4\pi D\, (C/N_0)^{2/3} z^{-1/3}. \tag{10}$$

Comparison with the results of the numerical calculation performed by Howell ([3]) shows that the variation of supersaturation with height is described quite satisfactorily, for $z < z_1$, by formulas (7) and (9), and for $z > z_1$ by (7) and (10). For $N_0 = 100$ cm$^{-3}$, $w = 600$ cm/sec, and the previously chosen values of $C$ and $D$, $z_1 \simeq 5 \cdot 10^3$ cm.

Allowance for coagulation will change the behavior of the supersaturation at large values of $z$. We shall assume that in this region $c \ll Cz$. Then, to determine the supersaturation in this region, we have the following linear equation:

$$dc/dz + kz^{1/3}e^{-\lambda z^2}c = C, \tag{11}$$

where

$$k = 4\pi D N_0^{2/3} C^{1/3} w^{-1}, \qquad \lambda = bC/2w\rho.$$

Since for $C = 0$ one must have $c \equiv 0$, it is necessary to choose the particular solution:

$$c(z) = C \int_0^z \exp\left\{-k \int_x^z t^{1/3} e^{-\lambda t^2}\, dt\right\} dx. \tag{12}$$

An approximate evaluation of the last integral, with an error not exceeding 20%, on the interval $5 \cdot 10^4$ cm $< z < 2 \cdot 10^5$ cm, gives

$$c(z) \simeq Cz\sqrt{\pi} e^{h^2} \operatorname{erfc} h \left[4h\left(\lambda z^2 - \frac{1}{6}\right)\right]^{-1}, \tag{13}$$

where

$$4h^2\left(\lambda z^2 - \frac{1}{6}\right) = kz^{4/3}e^{-\lambda z^2}. \tag{14}$$

From the results obtained it follows that the decrease in the number of drops in the ascending current as a result of coagulation leads to an increase of supersaturation with height, and at $z \simeq 2$ km $V(z)$ reaches its limiting value, equal to $\sim 3 \cdot 10^{-6}$. In calculating the growth of drops for $z > z_2$, the influence of condensation may be neglected.

The further growth of the rising drop, whose radius $r(t)$ corresponds to the mean volume of drops at height $z > z_2$, may be computed from the formula

$$r(t) = r(z_2) \exp\{0.33\, bV_2(t - t_2)\}, \tag{15}$$

where $V_2$ is the total volume of the liquid phase in 1 $cm^3$ at height $z_2$, $b = 6 \cdot 10^3$ $sec^{-1}$, and $t - t_2$ is the time of ascent of a drop from height $z_2$ to height $z$.

The growth of a falling drop is determined mainly by collisions with the drops of the ascending flow. If the fall velocity of a large drop of radius $R$ is taken to be $\Omega\sqrt{R}$, where $\Omega \simeq 2 \cdot 10^3$ $cm^{1/2}$/sec, then the equation for the change of $R$ will have the form

$$4R^2 \frac{dR}{dt} = \Omega \int_0^{\frac{4}{3}\pi R^3} (R + r)^2 (\sqrt{R} - \sqrt{r})\, n(v, t')\, dv, \tag{16}$$

where $n(v, t')$ is the number of drops of volume $v$ at the time $t'$ elapsed from the beginning of the ascent of the drops from the base of the cloud to the height to which, at the moment $t$, the drop of radius $R$ has descended.

Let us note that this equation is justified only for $R \gg r$, since it does not take into account the braking of the drop due to the acquisition of momentum from the drops of the ascending flow. But for $R \gg r$, equation (16) is substantially simplified:

$$\Omega\sqrt{R}V_2 = 4dR/dt. \tag{17}$$

The solution of this equation is obvious:

$$R(t) = \left[\frac{w}{\Omega} + \frac{\Omega V_2}{8}(t - t_h)\right]^2, \tag{18}$$

where $t_h$ is the moment at which the drop begins to descend.

Theoretical estimates of $R_m \sim 0.3$ cm, the radius of a drop breaking up in a turbulent flow of ascending air with the formation of a large number of smaller drops with mean radius $r_k \sim 0.01$ cm, were presented in the work of V. G. Levich ([4]). If the division of large drops occurs at the same height where the radius of the drop corresponding to the mean volume is equal to $r_k$, then the chain process predicted by Langmuir is possible.

Since the velocity of a drop may be considered equal to

$$\dot{z} = w - \Omega\sqrt{r}, \tag{19}$$

then, using formula (15), one can compute to what height a drop rises during the growth of its radius from $r_k$ to $r_h$, and also to what height the drop descends, in accordance with formulas (18) and (19), during the growth of the drop from radius $r_h$ to $R_m$. From this we obtain the equation which the velocity of the ascending air flow must satisfy for the formation of a chain process:

$$2b(\sqrt{R_m} - w/\Omega)^2 = 3w\left(\ln\frac{w}{\Omega\sqrt{r_k}} - 1\right) + 3\Omega\sqrt{r_k}. \tag{20}$$

For $R_m = 0.3$ cm and $r_k = 0.01$ cm, we obtain $w = 600$ cm/sec. For $V = 3\cdot 10^{-6}$, we obtain the dimensions of the cycle $z_h - z_m \simeq 1$ km, and the period of the cycle about 700 sec.

Let $N_k$ be the initial number of drops of radius $r_k$ per unit volume at the height where the breakup of the large drops occurs. Let $f$ be the probability of preservation of the water contained in a drop during the cycle; then the number of falling drops at the end of the $n$-th cycle will be

$$N_n = N_k f \frac{1-f^n}{1-f}\left(\frac{r_k}{r_h}\right)^3. \tag{21}$$

The criterion for raining out, i.e. the occurrence of a shower from the cloud, is the equality of the dynamic pressure created by the falling drops to the dynamic pressure of the ascending air flow of density $\rho_0$:

$$N_k f \frac{1-f^n}{1-f}\left(\frac{r_k}{r_h}\right)^3 \rho\Omega^2 R_m \simeq \rho_0 w^2. \tag{22}$$

Hence, for a known value one can estimate the number of cycles necessary for the formation of a shower.

For $R_m = 0.3$ cm, $r_k = 0.01$ cm, $w = 600$ cm/sec, a shower is possible only when $f \gtrsim 0.25$. Since the formation of a shower may require the completion of several cycles, it is necessary to consider the question of their stability, i.e., the possibility of moisture leaving the cycle.

Let two drops of identical radius be formed at heights whose difference is $\Delta z$. The coagulation growth of each of the drops can be expressed by the formula

$$\left(\frac{dv}{dz}\right)_{\text{coag}} \simeq bv\int_0^v n(u,z)\,u\,du. \tag{23}$$

It is obvious that as $z$ increases, because of coagulation the volume of moisture contained in relatively small drops decreases, which gives the impression of an absolutely unstable cycle, since in this case, in order to grow to $r_h$, the higher drop must travel a longer path. However, the lower-lying drop receives, in 1 sec, an excess impulse from coagulation as compared with the upper drop. Taking formula (27) into account, we obtain the following equation of motion:

$$\ddot{z} + b\dot{z}\int_0^v n(v',z)\,v'\,dv' \simeq 2\cdot 10^{-4} r^{-1}(w-\dot{z})^2 - g +$$

$$+\,b\int_0^v n(v',z)\,v'(w-\Omega\sqrt{r'})\,dv'. \tag{24}$$

An analogous equation is valid for a drop of the same volume at height $z+\Delta z$. Taking $\Delta z$ to be small, we arrive at the following equation for determining $\Delta z$:

$$\frac{d^2}{dt^2}\Delta z+\left[4\cdot 10^{-4}r^{-1}(w-\dot z)+b\int_0^v nv'\,dv'\right]\frac{d}{dt}\Delta z+$$

$$+\,b\int_0^v\left(-\frac{\partial n}{\partial z}\right)v'(w-\dot z-\Omega\sqrt{r'})\,dv'\,\Delta z=0. \tag{25}$$

The study of the stability of the trivial solution of the differential equation

$$y''(t)+p(t)y'(t)+q^2(t)y(t)=0 \tag{26}$$

by the change of variable $x=\int_0^t q(t)\,dt$ is reduced to the study of the stability of the trivial solution of a differential equation for which, as is known ([5]) from the analysis of the equivalent system of first-order differential equations, it is necessary, in order to ensure stability of the solution, to require that the inequality

$$p(t)+\frac{1}{2}\,d/dt\ln q^2(t)\geq 0. \tag{27}$$

Approximate estimates show that, as applied to equation (25), this inequality is satisfied.

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*Note: Figure translations are in progress. See original paper for figures.*