# ON A CLASS OF TANGENTIALLY DEGENERATE SURFACES

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## Abstract

## Full Text

## MATHEMATICS

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# ON A CLASS OF TANGENTIALLY DEGENERATE SURFACES

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1. In paper (1) we defined the focal images of a tangentially degenerate surface of rank $r$ and studied the structure of such a surface in the case when its focal images have no multiple components. In the present paper we consider surfaces of rank $r$ for which only one of the two focal images has no multiple components, while allowing their existence in the second image. We study the structure of the focal images of such surfaces and their intrinsic structure. The existence of surfaces of the type under consideration is also proved.
2. Let $V_{n,r}$ be an $n$-dimensional surface of rank $r$ in the projective space $P_N$. Its tangent plane $E_n$ depends on $r$ parameters and remains fixed when a point of the surface is displaced along its plane generator $E_m$, where $m = n - r$. The surface $V_{n,r}$ is an $r$-parameter family of plane generators $E_m$.

Associate with the surface $V_{n,r}$ a moving frame whose points $M_0, \dots, M_m$ lie in its generator $E_m$, $M_0, \dots, M_n$ in its tangent plane $E_n$, and the remaining points complete this system to a full frame of the space $P_N$. The equations of an infinitesimal displacement of this frame may be written in the form

$$dM_\xi = \omega_\xi^\eta M_\eta, \qquad \xi, \eta = 0, 1, \dots, N,$$

where

$$\omega_0^\alpha = 0, \qquad \omega_i^\alpha = 0 \tag{1}$$

for $i = 1, \dots, m,\ \alpha = n + 1, \dots, N$.

Assuming that the point $M_0$ does not lie on the focal image of the generator $E_m$, we may take the forms $\omega_0^p = \omega^p$ $(p = m + 1, \dots, n)$ as the basic forms of the surface $V_{n,r}$. They are linear combinations of the differentials of the independent variables $u^p$, on which the generator $E_m$ and the tangent plane $E_n$ of the surface $V_{n,r}$ depend. The forms $\omega_i^p$, which together with the forms $\omega_0^p$ determine the displacement of the plane generator $E_m$ of the surface $V_{n,r}$, and the forms $\omega_p^\alpha$, which determine the displacement of its tangent plane $E_n$, are expressed through the basic forms $\omega^p$ in the form

$$\omega_i^p = a_{iq}^p \omega^q, \qquad \omega_p^\alpha = b_{pq}^\alpha \omega^q. \tag{2}$$

As was shown in our paper ($^1$), the coefficients $a_{iq}^p$ and $b_{pq}^\alpha$ are connected by the relations

$$b_{pq}^\alpha = b_{qp}^\alpha, \qquad a_{ip}^s b_{sq}^\alpha = a_{iq}^s b_{sp}^\alpha,$$

where here and below the indices run through the following values: $i, j = 1, \dots, m$; $p, q, s = m+1, \dots, n$; $\alpha, \beta = n+1, \dots, N$. If we introduce the matrix notation $A_i = \|a_{ip}^q\|$ and $B^\alpha = \|b_{pq}^\alpha\|$, then the latter relations may

can be written in the form

$$(B^\alpha)^* = B^\alpha, \qquad (B^\alpha A_i)^* = B^\alpha A_i, \tag{3}$$

where the asterisk denotes transposition of the corresponding matrix.

If we denote by $X = x^0 M_0 + x_i^{iM}$ an arbitrary point of the generator $E_m$ of the surface $V_{n,r}$, and by $\Xi = \xi_\alpha x^\alpha = 0$ an arbitrary hyperplane passing through its tangent plane $E_n$, then the equation of the focal surface $F$ of the generator $E_m$ is written in the form

$$|\, x^0 E + x_i^{iA} \,| = 0,$$

where $E = \|\delta_\rho^\beta\|$ is the identity matrix, and the equation of the focal cone $\Phi$, whose vertex is the plane $E_n$, is written in the form

$$|\, \xi_\alpha B^\alpha \,| = 0.$$

The surface $F$ is an algebraic surface of order $r$, and the cone $\Phi$ is an algebraic surface of class $r$.

3. Using conditions (3) in a manner similar to what we did in paper ($^1$) for the surface $V_{n,r}$, whose focal images do not contain multiple components, we obtain in the case under consideration the following theorem:

**Theorem 1a.** *If on the surface $V_{n,r}$, for some fixed value of the variables $u^p$, the focal cone $\Phi$, having as its vertex the plane $E_n$, has no multiple components, while the focal surface $F$ lying in the plane $E_m$ has a unique t-fold component, then the matrices $A_i$ and $B^\alpha$ of the surface $V_{n,r}$ at this value of the parameters $u^p$ can be simultaneously reduced to the form*

$$A_i = \left\| \begin{matrix} a_i^{m+1} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & a_i^{n-t} & 0 \\ 0 & \dots & 0 & \widetilde{A}_i \end{matrix} \right\|, \qquad B^\alpha = \left\| \begin{matrix} b^\alpha_{m+1} & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & b^\alpha_{n-t} & 0 \\ 0 & \dots & 0 & \widetilde{B}^\alpha \end{matrix} \right\|, \tag{4}$$

*where* $\widetilde{A}_i$ *are scalar matrices of order* $t$*, and* $\widetilde{B}^\alpha$ *are arbitrary symmetric matrices of the same order.*

From this theorem there immediately follows the following theorem, describing the structure of the focal images of the surface $V_{n,r}$ under consideration:

**Theorem 2a.** *If, for some value of the variables* $u^p$*, the conditions of Theorem 1a are satisfied on the surface* $V_{n,r}$*, then at this value of* $u^p$ *its focal surface* $F$ *decomposes into* $r-t+1$ *distinct planes of dimension* $m-1$*, one of which is* $t$*-fold, while the focal cone* $\Phi$ *decomposes into* $r-t$ *distinct* $(N-n-2)$*-pencils of hyperplanes, each of which corresponds to a simple plane of the surface* $F$*, and a nondecomposable cone* $\widetilde{\Phi}$ *of class* $t$*, corresponding to the* $t$*-fold plane of the surface* $F$*.*

Suppose now that the condition of Theorem 1a is satisfied for all values of the variables $u^p$ from some domain $D$ of their variation. Investigation of the system of equations (2) and of its differential prolongations then makes it possible in this case to prove the following theorem, describing the structure of the surface $V_{n,r}$:

**Theorem 3a.** *If the conditions of Theorem 1a are satisfied for all values of the variables* $u^p$*, then the* $t$*-fold plane* $\widetilde{F}$ *of the focal surface* $F$ *for* $t \geq 2$ *depends on* $r-t$ *parameters and describes a surface* $V_{n-t-1,r-1}$*, while the surface* $V_{n,r}$ *decomposes into an* $(r-t)$*-parameter family of* $(m+t)$*-dimensional cones, each of which is a* $t$*-parameter family of planes* $E_m$ *passing through the fixed* $(m-1)$*-plane* $\widetilde{F}$*. On the other hand, the surface* $V_{n,r}$ *decomposes into a* $t$*-parameter*

ric family of $(n-t)$-dimensional surfaces $V_{n-t,r-t}$, whose focal images have no multiple components.

4. Analogously to what was indicated in Sec. 3, one can prove the following three dual theorems:

**Theorem 1b.** If on the surface $V_{n,r}$, for some fixed value of the variables $u^p$, the focal surface $F$ of the generator $E_m$ has no multiple components, while the corresponding focal cone $\Phi$ has a single $t$-fold component, then the matrices $A_i$ and $B^\alpha$ of the surface $V_{n,r}$, for this value of the parameters $u^p$, can be simultaneously reduced to the form (4), where now $\widetilde{A}$ are arbitrary square matrices of order $t$, and $B^\alpha$ are scalar matrices of the same order.

**Theorem 2b.** If, for some value of the parameters $u^p$ on the surface $V_{n,r}$, the conditions of Theorem 1b are satisfied, then for this value of $u^p$ its focal cone $\Phi$

decomposes into $r-t+1$ distinct $(N-n-2)$-webs of hyperplanes, one of which will be $t$-fold, and the focal surface $F$ decomposes into $r-t$ distinct $(m-1)$-planes, each of which corresponds to a simple web of the focal cone $\Phi$, and an irreducible surface $\widetilde{F}$ of order $t$, corresponding to the $t$-fold web of the cone $\Phi$.

**Theorem 3b.** If the conditions of Theorem 1b are satisfied for all values of the variables $u^p$ from some domain $D$, then the $t$-fold web of hyperplanes $\widetilde{\Phi}$ of the focal cone $\Phi$ for $t \geq 2$ depends on $r-t$ parameters. On these same parameters depends also the vertex of the web $\widetilde{\Phi}$, an $(n+1)$-plane $E_{n+1}$ passing through the tangent plane $E_n$. If these parameters are fixed, then the plane $E_n$ will move in the plane $E_{n+1}$ and envelope a hypersurface $V_{n,t} \subset E_{n+1}$ with $(n-t)$-dimensional plane generators. If, conversely, only these $r-t$ parameters are varied, then the planes $E_n$, tangent to $V_{n,r}$, will envelope a surface $V_{n,r-t}$ with $(m+t)$-dimensional plane generators, whose focal images contain no multiple components.

5. The surfaces $V_{n,r}$ described in Sec. 3 are determined by the system of equations (2), in which the matrices $A_i$ and $B^\alpha$ have the form (4). Suppose additionally that the net of developable surfaces on each of the submanifolds $V_{n-t,r-t}$, into which, by virtue of Theorem 3a, the surfaces $V_{n,r}$ decompose, is holonomic. Consider the matrix

$$A = \begin{Vmatrix} 1 & 1 & \cdots & 1 \\ a_1^{m+1} & a_1^{m+2} & \cdots & a_1^{n-t} \\ \cdots & \cdots & \cdots & \cdots \\ a_m^{m+1} & a_m^{m+2} & \cdots & a_m^{n-t} \end{Vmatrix},$$

formed from the eigenvalues of the matrices $A_i$. This matrix has $r-t$ columns. The investigation of the system of equations (4) now leads to the following existence theorem.

**Theorem 4a.** The surfaces $V_{n,r}$, described in Sec. 3, with a holonomic net of developable surfaces on the submanifolds $V_{n-t,r-t}$, for which the matrix $A$ has rank $r-t$, exist, and the arbitrariness of their existence is equal to $r-t$ functions of $t+1$ arguments. If, however, the matrix $A$ of these surfaces has rank less than $r-t$, and this rank is not lowered when any one of its columns is deleted from the matrix $A$, then such surfaces also exist, but the arbitrariness of their existence is equal to $N-t+1$ functions of $t$ arguments.

In exactly the same way one can formulate Theorem 4b on the existence of the surfaces $V_{n,r}$ described in Sec. 4.

We note that matrices analogous to the matrix $A$ were considered by us earlier in the paper ($^2$).

6. Let us dwell separately on the limiting cases of the surface $V_{n,r}$ of the type under consideration, when one of its focal images consists of a single $r$-fold component.

If each of the focal surfaces $F$ of the surface $V_{n,r}$ is an $r$-fold $(m-1)$-plane $E_{m-1}$, then this plane will remain fixed under all displacements of the generator $E_m$, and the surface $V_{n,r}$ becomes a cone whose vertex is this plane $E_{m-1}$. The focal cones $\Phi$ of such a surface $V_{n,r}$ will be irreducible cones of class $r$.

If each of the focal cones $\Phi$ of the surface $V_{n,r}$ is an $r$-fold $(N-n-2)$-bundle of hyperplanes, then the vertex of this bundle—the $(n+1)$-plane $E_{n+1}$, containing the tangent plane $E_n$—will remain fixed under all displacements of the plane $E_n$, and the surface $V_{n,r}$ becomes a hypersurface lying entirely in $E_{n+1}$. The focal surfaces $F$ of such a surface $V_{n,r}$ will be irreducible surfaces of order $r$.

The formulation of the existence theorems for these surfaces will differ somewhat from Theorems 4a and 4b. The point is that in this case one should apply the first part of the aforementioned theorems, but now $r - t = 0$. Therefore the arbitrariness in the existence of the surfaces under consideration will be equal to $N-n$ functions of $r$ arguments in the case when the focal surface $F$ has an $r$-fold component, and to $m+1$ functions of $r$ arguments in the case when the focal cone $\Phi$ has an $r$-fold component.

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*Note: Figure translations are in progress. See original paper for figures.*

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