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Abstract

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TWO VARIANTS OF THE CLASSICAL PREDICATE CALCULUS WITHOUT STRUCTURAL RULES OF INFERENCE

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§ 1. In this article the following logical symbolism is used. The expressions $t_1, t_2, \ldots$ are called **individual variables**.* The order in which they are listed here is called **alphabetical**. Expressions of the form P_j^0, where $j = 1, 2, \ldots$, are called **propositional variables**, and expressions of the form P_l^k, where $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$, are called **predicate variables**. Propositional variables and expressions of the form $P_l^k(z_1, \ldots, z_k)$, where $k, l \geq 1$ and $z_1, \ldots, z_k$ are arbitrary individual variables, are called **elementary formulas**.

Logical formulas (briefly, **formulas**) are defined by the following generating rules: 1) every elementary formula is called a formula; 2) if $\mathfrak{A}$ and $\mathfrak{B}$ are formulas, then the words $(\mathfrak{A} \supset \mathfrak{B})$, $(\mathfrak{A} \& \mathfrak{B})$, $(\mathfrak{A} \vee \mathfrak{B})$, and $\neg\mathfrak{A}$ are called formulas; 3) if x is an individual variable and $\mathfrak{A}$ is a formula, then the words $\forall x \mathfrak{A}$ and $\exists x \mathfrak{A}$ are called formulas.

Occurrences of individual variables in formulas are divided in the usual way into **free** and **bound** (see (1)). If x and y are individual variables and $\mathfrak{A}$ is a formula, then the expression $[\mathfrak{A}]_y^x$ will denote the formula obtained from $\mathfrak{A}$ as a result of substituting the variable y for all free occurrences of x in $\mathfrak{A}$. **Formula strings** are defined by the following generating rules: 1) the empty word is called a formula string; 2) if Σ is a formula string and $\mathfrak{A}$ is a formula, then the word $\Sigma\mathfrak{A}$ is called a formula string. Every expression of the form $\Gamma \to \Delta$, where Γ and Δ are formula strings, is called a **sequent**. Γ is called the **antecedent** of the sequent $\Gamma \to \Delta$, and Δ its **succedent**. The relation of **congruence**, introduced in § 33 of the book (1) for formulas, is naturally extended to sequents. Let us denote by φ the operation that transforms each sequent S into the sequent $\varphi(S)$, obtained from S by deleting, in all formulas of the sequent S, all occurrences of such quantifier complexes,** in whose scope there are no free occurrences of variables coinciding with the proper variables of these quantifier complexes. We shall say of sequents S_1 and S_2 that they are **almost congruent** if the sequents $\varphi(S_1)$ and $\varphi(S_2)$ are congruent. We shall say that a sequent S is a **pure sequent** if, first, no individual variable occurs in S both free and bound at the same time, and, second, the sequents S and $\varphi(S)$ are graphically equal.

Logical calculi whose derivable objects are sequents will be called **sequential calculi**. Let G' and G'' be two sequential calculi. We shall say that they are **equipollent** if two conditions are satisfied: 1) whatever

* In the symbolic language there will occur individual variables of only one sort.

** By quantifier complexes are meant words of the form $\forall x$ and $\exists x$, where x is some individual variable (x is called the proper variable of the quantifier complexes).

there was a sequent S_1, derivable in G', one can construct a sequent S_2, derivable in G'', such that S_1 and S_2 are almost congruent; condition 2) is obtained from 1) by permuting S_1 and S_2 with a simultaneous permutation of G' and G''.

§ 2. G. Gentzen in the work ([2]) constructed a sequent variant of the classical predicate calculus, which he called the calculus LK. In view of the difference between our logical symbolism and Gentzen' s logical symbolism, it will be more convenient for us to refer to the calculus G_1, described in § 77 of the book ([1]), which differs from LK only in technical details. The calculus G_1 is poorly adapted for a practically suitable algorithmization of the search for a logical derivation. Better adapted for this purpose is the calculus G_3, described in § 80 of the book just mentioned.

Below we construct new sequent variants E_0 and E'_0 of the classical predicate calculus, which have substantial differences from the calculi G_1 and G_3. Owing to their features, these calculi are better adapted to the search for derivations of sequents than G_1 and G_3. In the description below of the calculi E_0 and E'_0 there appears a number of syntactic (metamathematical) signs and expressions, to which the following meaning is assigned: $\mathfrak{A}$ and $\mathfrak{B}$ are arbitrary formulas; $x, y, z, z_1, z_2, ...$ are arbitrary individual variables; $\Gamma', \Gamma'', \Delta', \Delta''$ are arbitrary formula strings.

The calculus E_0 is given by the axiom schema

$$\Gamma''\mathfrak{A}\Gamma' \to \Delta'\mathfrak{A}\Delta''$$

and by the following rules of inference:

1. $$\frac{\mathfrak{A}\Gamma' \to \Delta'\mathfrak{B}\Delta''}{\Gamma' \to \Delta'(\mathfrak{A} \supset \mathfrak{B})\Delta''}$$

2. $$\frac{\Gamma'\Gamma'' \to \Delta'\mathfrak{A};\ \Gamma'\mathfrak{B}\Gamma'' \to \Delta'}{\Gamma'(\mathfrak{A} \supset \mathfrak{B})\Gamma'' \to \Delta'}$$

3. $$\frac{\Gamma' \to \Delta'\mathfrak{A}\Delta'';\ \Gamma' \to \Delta'\mathfrak{B}\Delta''}{\Gamma' \to \Delta'(\mathfrak{A}\&\mathfrak{B})\Delta''}$$

4. $$\frac{\Gamma'\mathfrak{A}\mathfrak{B}\Gamma'' \to \Delta'}{\Gamma'(\mathfrak{A}\&\mathfrak{B})\Gamma'' \to \Delta'}$$

5. $$\frac{\Gamma' \to \Delta'\mathfrak{A}\mathfrak{B}\Delta''}{\Gamma' \to \Delta'(\mathfrak{A} \vee \mathfrak{B})\Delta''}$$

6. $$\frac{\Gamma'\mathfrak{A}\Gamma'' \to \Delta';\ \Gamma'\mathfrak{B}\Gamma'' \to \Delta'}{\Gamma'(\mathfrak{A} \vee \mathfrak{B})\Gamma'' \to \Delta'}$$

7. $$\frac{\mathfrak{A}\Gamma' \to \Delta'\Delta''}{\Gamma' \to \Delta'\neg\mathfrak{A}\Delta''}$$

8. $$\frac{\Gamma'\Gamma'' \to \Delta'\mathfrak{A}}{\Gamma'\neg\mathfrak{A}\Gamma'' \to \Delta'}$$

9. $$\frac{\begin{array}{c}\Gamma' \to \Delta'\mathfrak{A}\Delta'' \\ x \text{ and } y \text{ satisfy condition } (*)\end{array}}{\Gamma' \to \Delta'\forall x\,[\mathfrak{A}]^y_x\Delta''}$$

10. $$\frac{\begin{array}{c}\Gamma'[\mathfrak{A}]^x_z\forall x\mathfrak{A}\Gamma'' \to \Delta' \\ x \text{ and } z \text{ satisfy condition } (**)\end{array}}{\Gamma'\forall x\mathfrak{A}\Gamma'' \to \Delta'}$$

11. $$\frac{\begin{array}{c}\Gamma' \to \Delta'[\mathfrak{A}]^x_z\exists x\mathfrak{A}\Delta'' \\ x \text{ and } z \text{ satisfy condition } (**)\end{array}}{\Gamma' \to \Delta'\exists x\mathfrak{A}\Delta''}$$

12. $$\frac{\begin{array}{c}\Gamma'\mathfrak{A}\Gamma'' \to \Delta' \\ x \text{ and } y \text{ satisfy condition } (*)\end{array}}{\Gamma'\exists x\,[\mathfrak{A}]^y_x\Gamma'' \to \Delta'}$$

Condition $(*)$: x does not occur in $\mathfrak{A}$; y occurs in the distinguished occurrence of $\mathfrak{A}$, and moreover only freely, and does not occur in the other occurrences of formulas in the premise*.

Condition $(**)$: the variable z is free for x in $\mathfrak{A}$ (see § 18 of the book ([1])) and either z is one of the individual variables occurring freely in S, where S is the sequent obtained from the premise of the rule of inference by deleting the distinguished occurrence of the formula $[\mathfrak{A}]^x_z$, or S has no free occurrences of individual variables and z is an arbitrary individual variable not occurring in S.

The calculus E_0' is obtained from the calculus E_0 by replacing the rules of inference

* It follows from condition $(*)$ that in the premise of this rule the formula $\mathfrak{A}$ occurs only once.

10 and 11, respectively, with the following rules of inference:

10′. $$\frac{\Gamma'[\mathfrak{A}]^x_{z_1} \ldots [\mathfrak{A}]^x_{z_n}\ \forall x\mathfrak{A}\Gamma'' \to \Delta', \quad x, z_1, \ldots, z_n \text{ satisfy condition } (***)}{\Gamma'\forall x\mathfrak{A}\Gamma'' \to \Delta'}$$

$$11'. \quad \frac{\Gamma' \to \Delta'[\mathfrak{A}]^x_{z_1} \dots [\mathfrak{A}]^x_{z_n} \exists x \mathfrak{A} \Delta'', \quad x, z_1, \dots, z_n \text{ satisfy condition } (***)}{\Gamma' \to \Delta' \exists x \mathfrak{A} \Delta''}$$

Condition $(***)$: a) the variables $z_1, \dots, z_n$ are pairwise distinct and each of these variables is free for x in $\mathfrak{A}$; b) either the list $z_1, \dots, z_n$ coincides with the complete list of object variables occurring freely in the sequent S obtained from the premise by deleting the indicated occurrences of the formulas $[\mathfrak{A}]^x_{z_1}, \dots, [\mathfrak{A}]^x_{z_n}$, or there are no free occurrences of object variables in S, $n = 1$, and z_1 does not occur in S.

§ 3. **Theorem 1.** *The calculi* $\mathbf{E}_0$ *and* $\mathbf{G}_1$ *are equipollent. Moreover, if* S *is a pure sequent, then it is derivable in* $\mathbf{E}_0$ *if and only if it is derivable in* $\mathbf{G}_1$.

The proof consists in indicating a method for transforming any derivation of some sequent in one calculus into a derivation of a sequent almost congruent to it in the other calculus. The transformation of a derivation in the calculus $\mathbf{E}_0$ into a derivation of the corresponding sequent in the calculus $\mathbf{G}_1$ is carried out without essential difficulties. The transformation of a derivation in the calculus $\mathbf{G}_1$ into a derivation of the corresponding sequent in the calculus $\mathbf{E}_0$ is carried out in several stages connected with the introduction of auxiliary calculi.

Theorem 2. *The calculi* $\mathbf{E}'_0$ *and* $\mathbf{G}_1$ *are equipollent. Moreover, if* S *is a pure sequent, then it is derivable in* $\mathbf{E}'_0$ *if and only if it is derivable in* $\mathbf{G}_1$.

This theorem is proved by establishing the equipollence of the calculi $\mathbf{E}'_0$ and $\mathbf{E}_0$.

Let us note the following two features of the calculi $\mathbf{E}_0$ and $\mathbf{E}'_0$: 1) in the calculi $\mathbf{E}_0$ and $\mathbf{E}'_0$ the structural rules are absent; 2) if some sequent S_1 is obtained from a sequent S_2 (from sequents S_2 and S_3) as a result of a single application of some rule of inference of the calculus $\mathbf{E}_0$, then S_1 is derivable in $\mathbf{E}_0$ only when S_2 is derivable in this calculus (respectively, S_2 and S_3 are derivable); the same is true for the calculus $\mathbf{E}'_0$.

Remark. J. Herbrand in ([3]) proved the possibility of transforming any derivation in the classical Hilbert-type predicate calculus into such a derivation in which, when applying quantifier rules, fairly strict restrictions on object variables are satisfied. The restrictions on object variables in the quantifier rules of the calculi $\mathbf{E}_0$ and $\mathbf{E}'_0$ are analogous to the Herbrand restrictions just mentioned. The calculus $\mathbf{E}'_0$ has features similar to the method of semantic tableaux proposed by E. Beth in ([4]).

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Note: Figure translations are in progress. See original paper for figures.