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Abstract

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HOROSPHERICAL RADIAL PARTS OF LAPLACE OPERATORS ON SYMMETRIC SPACES

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Let G be the group of real matrices of order n with determinant one, and let U be its maximal compact subgroup. We shall assume that U consists of all orthogonal matrices of order n with determinant one. Denote by $\mathcal{E}$ the homogeneous space G/U. Consider the group Z, consisting of all triangular matrices for which all entries below the main diagonal are zero, and all diagonal entries are equal to one. The trajectories of the group Z in the space $\mathcal{E}$, by analogy with ([1]), will be called horospheres. Trajectories of groups analogous to the group Z (horospheres) are of great importance in representation theory. This fact was repeatedly emphasized by I. M. Gel' fand.

As is known, in the theory of functions on a symmetric space an important role is played by the so-called Laplace operators, i.e., differential operators commuting with all motions of the space. For some symmetric spaces in ([2]), the restrictions of these operators to the set of functions constant on the trajectories of the stationary subgroup were computed (the so-called radial parts of the Laplace operators). For arbitrary symmetric spaces the radial parts of the Laplace operators have not been computed, and examples show that these operators have a complicated structure.

In the present note we compute the restrictions of the Laplace operators to functions constant on the trajectories of the group Z. We call the corresponding operators the horospherical radial part of the Laplace operators. It follows from Theorem 1 that the horospherical radial parts of the Laplace operators have a simple structure. Horospherical radial parts turn out to be useful in representation theory, as well as in the theory of automorphic and harmonic functions.

Let $f(x)$ $(x \in \mathcal{E})$ be some function on $\mathcal{E}$. By $T_g f$ we shall denote the function obtained from the function f by the shift corresponding to the motion g, namely

$$T_g f(x) = f(g^{-1}x).$$

(By $g^{-1}x$ we denote the point of the space $\mathcal{E}$ obtained from the point $x \in \mathcal{E}$ by the motion g^{-1}.) Consider the space R of functions on $\mathcal{E}$ that are constant on horospheres, i.e., functions f satisfying the equality $T_z f = f$ for any matrix

$z \in Z$. It is known that every matrix $g \in G$ can be represented in the form $g = zhu$, where $z \in Z$, $u \in U$, and h is a diagonal matrix all of whose diagonal entries are positive. Hence it follows that every point $x \in \mathcal{E}$ can be represented in the form $x = zhx_0$, where x_0 is a point of $\mathcal{E}$ whose stationary subgroup is U. Therefore functions f from the space R may be regarded as functions on the group H of diagonal matrices with positive diagonal entries. Denote by $t = (t^1, \dots, t^n)$ the logarithms of the diagonal entries of the matrix h. Then $t^1 + \dots + t^n = 0$, and hence a function constant on horospheres may be regarded as a function on the hyperplane L defined by the equation

$$\sum_1^n t^i = 0.$$

The variables $t^1, \dots, t^{n-1}$ determine certain coordinates on the hyperplane L. Therefore a function f from R may be regarded as a function

$$f = f(t^1, \dots, t^{n-1}).$$

Let Δ be an arbitrary Laplace operator on the space $\mathcal{E}$ (see, for example, (2)), i.e., a differential operator commuting with any operator T_g $(g \in G)$. If f is a function constant on horospheres, then Δf is also a function constant on horospheres. Consequently, every Laplace operator Δ induces a certain differential operator $\overset{0}{\Delta}$ on the set of functions $f = f(t^1, \dots, t^{n-1})$ that are constant on horospheres. It is natural to call the operator $\overset{0}{\Delta}$ the horospherical radial part of the operator Δ. Our goal is to compute the operators $\overset{0}{\Delta}$.

Theorem 1. *In the coordinates $t^1, \dots, t^{n-1}$, the operator $\overset{0}{\Delta}$ is an operator with constant coefficients.*

Proof. Note that if q is an arbitrary diagonal matrix and $z \in Z$, then $qzq^{-1} \in Z$. Hence it follows that the operator T_q maps a function belonging to R again into a function belonging to R, and induces a certain operator $\overset{0}{T}_q$ on the set of functions constant on horospheres. Therefore the operator $\overset{0}{\Delta}$ must commute with all operators $\overset{0}{T}_q$. Let $\tau = (\tau^1, \dots, \tau^n)$ be the logarithms of the diagonal elements of the matrix q $(q \in H)$. Then it is easy to see that $\overset{0}{T}_q f(t) = f(t - \tau)$, and every differential operator commuting with the operators $f(t) \to f(t - \tau)$ is an operator with constant coefficients. The theorem is proved.

Let $t = (t^1, \dots, t^n)$ and $\tau = (\tau^1, \dots, \tau^n)$ be two vectors on the hyperplane L. We introduce on them the scalar product by the formula

$$(t, \tau) = \sum_1^n t^i \tau^i.$$

On the hyperplane L there acts the group S of transformations consisting of all possible permutations of the variables $t^1, \ldots, t^n$. Theorem 2, proved below, gives the explicit form of the operators $\overset{0}{\Delta}$ and at the same time shows what symmetries for the operators $\overset{0}{\Delta}$ are caused by the presence of the group S.

Theorem 2. a) *Every operator* $\overset{0}{\Delta}$ *has the form*

$$\overset{0}{\Delta} = e^{(\Lambda,t)} D e^{-(\Lambda,t)}, \tag{1}$$

where Λ *is the vector determined by the equality*

$$(\Lambda, t) = \frac{1}{2} \sum_{i<j} (t^i - t^j),$$

D is an arbitrary differential operator with constant coefficients (in the coordinates $t^1, \ldots, t^{n-1}$), acting in the space of functions defined on the hyperplane L, and remaining invariant under all transformations of the group S.

b) *Conversely, every operator* $\overset{0}{\Delta}$ *given by formula (1) is the horospherical radial part of some Laplace operator.*

Proof. To each transformation from the group S there corresponds a certain orthogonal matrix $s \in U$ such that the transformation $h \to shs^{-1}$ $(h \in H)$ realizes the corresponding permutation of the diagonal elements of the matrix h. If a function $f(x)$ $(x \in \mathcal{E})$ is constant on the trajectories of the group Z, then the function $T_s f$ is constant on the trajectories of the group $Z_s = sZs^{-1}$. Denote by R_s the set of functions constant on the trajectories of the group Z_s. These functions may also be regarded as functions on the group H, and the horospherical (relative to the group Z_s) radial part

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* Theorem 1 once again emphasizes the important role of the trajectories of the group Z.

$\overset{0}{\Delta}_s$ of the Laplace operator Δ is related to the operator $\overset{0}{\Delta}$ by the relation

$$\overset{0}{\Delta}_s = \overset{0}{T}_s \overset{0}{\Delta} \overset{0}{T}_s^{-1},$$

where by $\overset{0}{T}_s$ we have denoted the operator in the space of functions on H corresponding according to the formula $\overset{0}{T}_s f(h) = f(s^{-1}hs)$ $(h \in H)$.

Let now $f(x)$ $(x \in \mathcal{E})$ be an arbitrary infinitely differentiable function from the space R such that the corresponding function $f(h) = f(hx_0)$ $(h \in H)$ on the group H is finite. Averaging the function $f(x)$ over the trajectories of the group Z_s, we obtain a function $\varphi(x)$ from the space R_s. More precisely, put $\widetilde{Z} = Z \cap Z_s$, and let $Z = Z_s/\widetilde{Z}$. The function $T_{z_s} f(x)$ ($z_s \in Z_s$), as a function of z_s, is constant on the cosets modulo the subgroup $\widetilde{Z}$, and therefore it may be regarded as a function $f(x, \zeta)$, where $\zeta \in Z$. It is easy to prove that if x runs through an arbitrary compact set in $\mathcal{E}$, then $f(x, \zeta)$, as a function of ζ, is finite. The function $\varphi(x) = \int f(x, \zeta)\, d\zeta$, where $d\zeta$ is the invariant measure on the homogeneous space Z, evidently belongs to R_s. The corresponding function $\varphi(h)$ on the group H is defined by the equality $\varphi(h) = \varphi(hx_0) = \int f(hx_0, \zeta)\, d\zeta$. Put $z_s^{-1}x_0 = zqx_0$, where $z_s \in Z_s$, $z \in Z$, $q \in H$. The matrix q, as a function of z_s, is constant on the cosets modulo the subgroup $\widetilde{Z}$, and therefore $q = q(\zeta)$ ($\zeta \in Z$). Further we have

$$hz_s^{-1}x_0 = hz_s^{-1}h^{-1}hx_0 = hz_s^{-1}hqx_0.$$

Hence it is seen that $f(hx_0, \zeta_1) = f(hq(\zeta) \cdot x_0) = f(hq(\zeta))$, where the transformation $\zeta \to \zeta_1$ is the transformation of the space Z induced by the isomorphism $z_s \to hz_{sh}^{-1}$ of the group Z_s. Therefore

$$\varphi(h) = \int f(hq(\zeta))\, d\zeta_1.$$

Let $t = (t^1, \dots, t^n)$ be the logarithms of the diagonal elements of the matrix h. Then it is easy to compute that the measures $d\zeta$ and $d\zeta_1$ are related by

$$d\zeta_1 = \exp(M, t)\, d\zeta,$$

where $M = \Lambda_s - \Lambda$, and Λ_s is the vector obtained from the vector Λ by the corresponding transformation of the group S. Thus,

$$\varphi(h) = \int \exp(M, t)\, f(hq(\zeta))\, d\zeta.$$

Therefore,

$$\overset{0}{\Delta}_s \varphi(h) = \int \overset{0}{\Delta}_s \exp(M, t) f(hq(\zeta))\, d\zeta.$$

On the other hand, by virtue of the commutativity of the operators Δ and T_{z_s}, we obtain

$$\overset{0}{\Delta}_s \varphi(h) = \int \exp(M, t)\, \overset{0}{\Delta} f(hq(\zeta))\, d\zeta.$$

By virtue of the arbitrariness of the function f, we find that

$$\overset{0}{\Delta}_s \exp(\Lambda_s - \Lambda, t) f(hq(\zeta)) = \exp(\Lambda_s - \Lambda, t)\, \overset{0}{\Delta} f(hq(\zeta)).$$

Hence, using (2), we obtain that the operator

$$D = e^{-(\Lambda,t)} \overset{0}{\Delta} e^{(\Lambda,t)}$$

is invariant under all transformations of the group S. This proves part a) of Theorem 2. Part b) follows at once from the fact that to every invariant symmetric tensor $g^{i_1 \dots i_k}$ on the space $\mathcal{E}$ there corresponds a Laplace operator of order k, the coefficients at whose highest derivatives are equal to the corresponding components of the tensor ([2]).

Theorems 1 and 2 are valid for any homogeneous space $\mathcal{E} = G/U$ whose group of motions G is semisimple, and whose stationary subgroup U coincides with a maximal compact subgroup of the group G. As is known, in this case the group G has an involutive automorphism $g \to g'$, the set of fixed points of which coincides with U. Denote by T the set of elements g of the group G such that $g' = g^{-1}$. The Cartan metric in G induces on $\mathcal{E}$ an invariant Riemannian metric with respect to which $\mathcal{E}$ is a symmetric space of nonpositive curvature. The group Z can be constructed as follows. Consider an arbitrary geodesic γ of the space $\mathcal{E}$, passing through the point x_0, whose stationary subgroup coincides with U. Every such geodesic is the trajectory of a one-parameter subgroup $h(t) \subset T$. If

γ is a geodesic in general position ([3]), then the group Z consists of all those elements $z \in G$ for which $\lim_{t \to +\infty} h^{-1}(t) z h(t) = e$, where e is the identity of the group G. The role of the group of diagonal matrices with positive diagonal elements is played by a maximal commutative subgroup $H \subset T$ which contains the subgroup $h(t)$. The role of the coordinates $t^1, \dots, t^{n-1}$ is played by the canonical coordinates in the group H. The group S is the Weyl group of the symmetric space $\mathcal{E}$, and the scalar product (t, τ) is the Cartan scalar product in the algebra of the group H. Finally, the vector Λ is equal to the half-sum of all positive roots (with respect to a suitable ordering) of the symmetric space $\mathcal{E}$.

We note that, in the case when γ is not a geodesic in general position, theorems analogous to Theorems 1 and 2 are also valid.

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Note: Figure translations are in progress. See original paper for figures.