

ON ONE TYPE OF RIBAUCCOUR TRANSFORMATIONS

In the present note one special type of Ribaucour transformation is considered.

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. A. TIKHONOV

ON ONE TYPE OF RIBAUCCOUR TRANSFORMATIONS

(Presented by Academician I. N. Vekua on 14 XII 1961)

In the present note one special type of Ribaucour transformation is considered.

1. Let us consider, in three-dimensional conformal space, a transformation of surfaces consisting in the passage from one sheet of the envelope of a congruence of spheres to the other sheet in such a way that on these surfaces the conformally asymptotic lines of Takasu, determined by the transforming congruence of spheres, correspond. It turns out that this transformation belongs to the class of Ribaucour transformations (R -transformations).

We shall call it an R^* -transformation. In posing the problem of finding all R^* -transformations of a given surface, suppose that the surface is referred to the canonical frame of Gaidelman ⁽¹⁾. This frame consists of two points A_0 and A_4 and three spheres S_1, S_2, S_3 (the symbols A and S denote, respectively, an analytic point and an analytic sphere), of which S_1 and S_2 are the tangent spheres of the lines of curvature, normal to the surface, while S_3 is the central sphere of the surface. The elements of the frame are connected by the relations:

$$\begin{aligned} (A_0A_0) &= (A_4A_4) = (A_0S_i) = (A_4S_i) = 0, \\ (A_0A_4) &= 1, \quad (S_iS_k) = \delta_{ik}, \quad i, k = 1, 2, 3. \end{aligned}$$

The formulas for the infinitesimal displacements of the canonical frame have the form

$$\begin{aligned} dA_0 &= \omega_0^0 A_0 + \omega^1 S_1 + \omega^2 S_2, \\ dS_1 &= \omega_1^0 A_0 + \omega^1 S_3 - \omega^1 A_4, \\ dS_2 &= \omega_2^0 A_0 - \omega^2 S_3 - \omega^2 A_4, \\ dS_3 &= \omega_3^0 A_0 - \omega^1 S_1 + \omega^2 S_2, \\ dA_4 &= -\omega_1^0 S_1 - \omega_2^0 S_2 - \omega_3^0 S_3 - \omega_0^0 A_4, \end{aligned}$$

where $[\omega^1 \omega^2] \neq 0$,

$$\omega_0^0 = b\omega^1 - c\omega^2, \quad \omega_1^0 = -(l + \frac{1}{2})\omega^1 + m\omega^2, \quad \omega_2^0 = k\omega^1 + (l - \frac{1}{2})\omega^2, \quad \omega_3^0 = b\omega^1 + c\omega^2.$$

The equality $k - m = 0$ characterizes isothermic surfaces, and the conditions $b = c = k = m = 0$, $l = \text{const}$ characterize Dupin cyclides.

Let

$$\bar{S} = \lambda(u, v)A_0 + S_3$$

be the transforming congruence of spheres of the surface (A_0) . Then the second sheet of the envelope is determined by the formula

$$P = \xi^0 A_0 + \xi^1 S_1 + \xi^2 S_2 - \lambda S_3 + A_4,$$

where

$$\xi^1 = -\frac{\lambda_1 + (\lambda + 1)b}{\lambda - 1}, \quad \xi^2 = -\frac{\lambda_2 - (\lambda - 1)c}{\lambda + 1}, \quad \xi^0 = -\frac{(\xi^1)^2 + (\xi^2)^2 + \lambda^2}{2},$$

and λ_1 and λ_2 are the covariant derivatives of the function λ with respect to the pair of Pfaffian forms ω^1, ω^2 . In the paper ⁽²⁾ it is shown that, in order that the surface

that the surface (P) be an R^* -transformation of the surface (A_0) , it is necessary and sufficient that the function λ satisfy the following equation in covariant derivatives (the equation of the R -transformation):

$$(1 - \lambda^2)\lambda_{12} + 2\lambda\lambda_1\lambda_2 + 2c(\lambda - 1)\lambda_1 + b(\lambda + 1)^2\lambda_2 + (1 - \lambda^2)\{(2bc + k + m)\lambda + k - m\} = 0. \quad (1)$$

Let us find the conditions which the function λ must satisfy in order that the surface (P) be an R^* -transformation of the surface (A_0) . The conformally asymptotic lines on the surface (A_0) are determined by the equation $(dA_0 d\bar{S}) = 0$, or

$$(\lambda - 1)(\omega^1)^2 + (\lambda + 1)(\omega^2)^2 = 0.$$

The conformally asymptotic lines on the surface (P) are determined by the equation $(dP d\bar{S}) = 0$, or

$$\alpha(\omega^1)^2 + \beta\omega^1\omega^2 + \gamma(\omega^2)^2 = 0,$$

in which α, β, γ have the values

$$\alpha = -\lambda_1 b - \xi^1(\lambda_1 + \lambda b + b) + (\lambda - 1)\xi_1^1 + (\lambda - 1)(l + 1/2) - \lambda b^2 + (\lambda - 1)\lambda + \xi^0(\lambda - 1),$$

$$\beta = (\lambda - 1)\xi_2^1 + (\lambda + 1)\xi_1^2 + \{(\lambda - 1)c - \lambda_2\}\xi^1 - \{(\lambda + 1)b - \lambda_1\}\xi^2 +$$

$$+\lambda_1 c - \lambda_2 b - (\lambda - 1)m - (\lambda + 1)k + 2\lambda bc,$$

$$\begin{aligned} \gamma = & \lambda_2 c - \xi^2(\lambda_2 - \lambda c + c) + (\lambda + 1)\xi_2^2 - (\lambda + 1)(l - 1/2) - \\ & - \lambda c^2 - (\lambda + 1)\lambda + \xi^0(\lambda + 1), \end{aligned}$$

where $\lambda_i, b_i, \xi_i^k, c_i$ denote, respectively, the covariant derivatives of the functions λ, b, ξ^k, c . For the covariant derivatives ξ_i^k we have the expressions

$$\begin{aligned} \xi_i^1 = & -\frac{(\lambda - 1)\lambda_{1i} - \lambda_1 \lambda_i - 2b\lambda_i + (\lambda^2 - 1)b_i}{(\lambda - 1)^2}, \\ \xi_i^2 = & -\frac{(\lambda - 1)\lambda_{2i} - \lambda_1 \lambda_i - 2c\lambda_i + (\lambda^2 - 1)c_i}{(\lambda + 1)^2}. \end{aligned}$$

Requiring the correspondence of the conformally asymptotic lines on the surfaces (A_0) and (P) , we arrive at two conditions $\beta = 0$, $(\lambda + 1)\alpha = (\lambda - 1)\gamma$. Substituting in these conditions the expressions for α, β , and γ given above, and taking into account the equalities $b_2 = 2bc + k$, $c_1 = -(2bc + m)$, $\lambda_{21} = \lambda_{12} - c\lambda_1 - b\lambda_2$, we obtain, for determining the function λ , the system of two equations in covariant derivatives

$$\begin{aligned} (1 - \lambda^2)\lambda_{12} + 2\lambda\lambda_1\lambda_2 + 2c(\lambda - 1)\lambda_1 + b(\lambda + 1)^2\lambda_2 + \\ + (1 - \lambda^2)\{(2bc + k + m)\lambda + k - m\} = 0, \end{aligned} \quad (2)$$

$$\begin{aligned} (\lambda^2 - 1)\{(\lambda - 1)\lambda_{22} - (\lambda + 1)\lambda_{11}\} + 2(\lambda + 1)^2(\lambda_1)^2 - 2(\lambda - 1)^2(\lambda_2)^2 + \\ + (\lambda + 1)(\lambda^2 + 6\lambda + 5)b\lambda_1 + (\lambda - 1)(\lambda^2 - 6\lambda + 5)c\lambda_2 + \mathcal{L} = 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{L} = & 2(\lambda^2 - 1)^2(\lambda + l) + 2(\lambda + 1)^3b^2 + 2(\lambda - 1)^3c^2 - \\ & - (\lambda + 1)^2(\lambda^2 - 1)b_1 - (\lambda - 1)^2(\lambda^2 - 1)c_2. \end{aligned}$$

The first equation of the system (2) coincides with the equation of the R -transformation (1), whence the theorem follows:

Theorem 1. Every R^* -transformation is a Ribaucour transformation.

The following theorem holds:

Theorem 2. *If the surface (P) is obtained from the surface (A_0) by an R^* -transformation, then the natural correspondence between the points of these surfaces is a conformal correspondence.*

The proof of this theorem is not difficult to obtain by using the basic equations of the conformal theory of congruences of spheres ⁽³⁾, and we shall omit it.

2. It is known that isothermic surfaces, and only they, admit an R -transformation by means of a congruence of their central spheres ⁽⁴⁾. It is natural to ask whether, in the class of isothermic surfaces, there exist surfaces that admit an R^* -transformation by means of a congruence of central spheres. Assuming that the second equation of the system is satisfied for $\lambda \equiv 0$, we arrive at the condition $b_1 + c_2 + 2(b^2 - c^2 + l) = 0$. It is not difficult to show that this condition, together with the condition $k - m = 0$, characterizes isothermic conformally minimal surfaces (minimal surfaces of spaces of constant curvature).

Thus, we have the theorem:

Theorem 3. *Isothermic conformally minimal surfaces, and only they, admit an R^* -transformation by means of a congruence of their central spheres.*

3. The system of equations (2) has the simplest form for Dupin cyclides. For these surfaces we have $D\omega^1 = 0$, $D\omega^2 = 0$ (D is the symbol of the exterior differential), and therefore we may put $\omega^1 = du$, $\omega^2 = dv$. Consequently, all covariant derivatives will be ordinary derivatives, and system (2) takes the form

$$(1 - \lambda^2)\lambda_{uv} + 2\lambda\lambda_u\lambda_v = 0,$$

$$(\lambda^2 - 1)\{(\lambda - 1)\lambda_{vv} - (\lambda + 1)\lambda_{uu}\} + 2(\lambda + 1)^2\lambda_u^2 -$$

$$-2(\lambda - 1)^2\lambda_v^2 + 2(\lambda^2 - 1)^2(\lambda + l) = 0.$$

Every Dupin cyclide admits an R^* -transformation. As an example one may indicate a two-parameter family of R^* -transformations corresponding to a function $\lambda(v)$, determined as the general solution of an ordinary differential equation of the second order:

$$(\lambda + 1)\lambda'' - 2\lambda'^2 + 2(\lambda + 1)^2(\lambda + l) = 0. \quad (3)$$

Equation (3) is integrated in elementary functions.

Voronezh State University

Received
12 XII 1961

CITED LITERATURE

1. R. M. Geidelman, *Construction of the conformal theory of a surface by the method of a moving frame*, Dissertation, Moscow City Pedagogical Institute, 1948.
2. V. A. Tikhonov, *Izv. Vyssh. uchebn. zaved.*, Mathematics, No. 3, 136 (1961).
3. R. M. Geidelman, DAN, **134**, No. 4 (1960).
4. W. Blaschke, *Vorlesungen über Differentialgeometrie*, 3, Berlin, 1929.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.