ON THE UNIQUENESS OF A MINIMAL SEMIREDUCIBLE DECOMPOSITION

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Abstract

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MATHEMATICS

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ON THE UNIQUENESS OF A MINIMAL SEMIREDUCIBLE DECOMPOSITION

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1. The study of a number of problems connected with isometric ([1]), projective ([2]), and conformal ([3]) transformations of Riemannian spaces V_n leads to the question of the uniqueness of the representation of the metric ds^2 in the form

$$ds^2 = ds_0^2(x^a) + \sigma(x^a)\, ds_1^2(x^\alpha) \tag{1}$$

$$(a = 1, \dots, q;\ \alpha = q+1, \dots, n).$$

A space V_n admitting at least one representation of ds^2 in the form (1) is called semireducible. The metric ds_0^2 is called the principal part of the semireducible decomposition (1), and ds_1^2 its supplementary part ([4]).

In a number of cases, besides ds_1^2, other supplementary metrics $ds_2^2, \dots, ds_p^2$ are separated from ds_0^2, and then the metric (1) is represented in the form

$$ds^2 = ds_0^2(x^a) + \sigma_1(x^a)\, ds_1^2(x^{\alpha_1}) + \dots + \sigma_p(x^a)\, ds_p^2(x^{\alpha_p}), \tag{2}$$

where no two of the functions $\sigma_1, \dots, \sigma_p$ are in a constant ratio, i.e. $\sigma_\nu/\sigma_\mu \neq \text{const}$ (otherwise the corresponding summands in (2) are combined).

If in (1) the metric ds_1^2, in its turn, is semireducible:

$$ds_1^2 = d\tau_0^2(x^{\alpha_1}) + \nu(x^{\alpha_1})\, d\tau_1^2(x^{\alpha_2}),$$

then the decomposition (1) is continued to the right, i.e. from it one obtains a new semireducible decomposition of the metric ds^2:

$$ds^2 = d\tilde{s}_0^2 + \tilde{\sigma}\, d\tau_1^2, \tag{3}$$

where

$$d\tilde{s}_0^2 = ds_0^2 + \sigma\, d\tau_0^2, \quad \tilde{\sigma} = \sigma\nu.$$

The inverse passage from (3) to (1) will be called a continuation of the semireducible metric ds^2 to the left. Obviously, under continuation to the right the dimension of the principal part increases, and under continuation to the left it decreases. Continuation to the right and to the left is defined analogously for the decomposition (2) as well.

In this note it is assumed that the space V_n is a proper Riemannian space ($ds^2 > 0$). Some results on the question of the uniqueness of the representation of a semireducible metric of a proper Riemannian space in the form (2) were obtained by the author in ([4]). Below more general results are established, the principal one being the theorem on the uniqueness of a minimal semireducible decomposition.

2. A symmetric tensor A_{ij} $(i, j = 1, \dots, n)$, defined with respect to the metric (1) by the form

$$dA^2 = A_{ij}\, dx^i dx^j = \sigma(x^a)\, ds_1^2(x^\alpha), \tag{4}$$

satisfies, together with the function $\psi = \ln \sigma$, the system of equations (4)

$$A_{ij,k} = -\frac{1}{2}(\psi_{,i} A_{jk} + \psi_{,j} A_{ik}), \tag{5}$$

$$A_{ik} A_j^k = A_{ij}. \tag{6}$$

Conversely, every nontrivial* solution (A_{ij}, ψ) of this system determines a semireducible decomposition (1), i.e., in some coordinate system (1) and (4) are satisfied. We note that for the metric (2) each summand $\sigma_\nu ds_\nu^2$ $(\nu = 1, \dots, p)$ determines a solution of the system (5), (6). Consequently, the question of the uniqueness of the representation of the metric ds^2 in the semireducible form (1) or (2) is reduced to the question of the number of solutions of this system.

If the semireducible decomposition (2) (or, in particular, (1)) admits a continuation to the right or to the left, then the tensor A_{ij} determined by one of the additional metrics of the continued decomposition and therefore satisfying the system (5), (6), will have, relative to (2), zero "mixed" components:

$$A_{\alpha\alpha_\nu} = 0, \qquad A_{\alpha_\nu \beta_\mu} = 0, \qquad (\nu \neq \mu). \tag{7}$$

As the following lemma shows, the converse assertion is also true; moreover, the second of conditions (7) is a consequence of the first.

Lemma 1. Every nontrivial solution (A_{ij}, ψ) of the system (5), (6) in the space (2), satisfying the condition $A_{\alpha\alpha_\nu} = 0$, determines either the semireducible decomposition (2) itself, or its continuation to the right or to the left.

3. It is not difficult to see that a metric of constant curvature can be represented in the form (1) and in the form (2) in infinitely many ways. One may only note that every such representation is a K-decomposition ([2]). The spaces V_n admitting at least one K-decomposition are called spaces $V(K)$. A geometric description of all K-decompositions in a space $V(K)$ and, in particular, in a space of constant curvature is given in ([5]).

In the majority of spaces $V(K)$ of nonconstant curvature there also exists an infinite set of K-decompositions; however, among them there is one which is uniquely determined—this is the so-called maximal K-decomposition, i.e., the one having the largest main part ds_0^2 in dimension. As was shown in ([6]), every other K-decomposition is obtained from the maximal one by continuation to the left. The question arises: how is every other semireducible decomposition (2) in a space $V(K)$, not a K-decomposition, related to the maximal K-decomposition? The answer is supplied by the theorem:

Theorem 1. Every semireducible decomposition (2) that is not a K-decomposition in a space $V(K)$ of nonconstant curvature is obtained from its maximal K-decomposition by continuation to the right.

Thus, the maximal K-decomposition is at the same time minimal with respect to any semireducible decomposition that is not a K-decomposition.

4. In a reducible space V_n it is natural to take as fundamental the complete reducible decomposition of its metric:

$$ds^2 = dt_1^2(x^{a_1}) + \cdots + dt_p^2(x^{a_p}), \tag{8}$$

where dt_ν^2 are irreducible metrics. Disregarding, in addition, the case of a space $V(0)$ considered above, suppose that all the metrics dt_ν^2 are not one-dimensional. In this case the decomposition (8) is minimal with respect to any decomposition of the form (2) in the same space. Indeed, the following theorem holds:

Theorem 2. Every semireducible decomposition (2) in a reducible V_n, not a space $V(0)$, is obtained from its complete reducible

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* That is, $A_{ij} \neq 0$ and does not coincide with the metric tensor.

of the decomposition (8) by extension to the right by means of a semi-reducible splitting of one of the metrics dt_ν^2.

5. Let us now consider a semi-reducible space (1), but one that is irreducible and is not $V(K)$. Investigating in such a space the system of equations (5) and the conditions for its integrability, we arrive at the lemma:

Lemma 2. Every tensor A_{ij} satisfying, together with some function ψ, system (5) in an irreducible space (1) that is not $V(K)$, obeys the condition $A_{aa} = 0$, if at least one of the following requirements, imposed on the principal part ds_0^2 of the metric (1), is satisfied:

1) ds_0^2 is a metric of constant curvature K;
2) ds_0^2 is the metric of a space $V(K)$ of nonconstant curvature;
3) ds_0^2 is a reducible metric;
4) ds_0^2 is a one-dimensional metric;
5) ds_0^2 is a semi-reducible metric with a one-dimensional principal part;
6) ds_0^2 is a non-one-dimensional and non-semi-reducible metric.

The geometric meaning of this lemma is that, under the indicated conditions, any other semi-reducible decomposition is obtained from (1) either by extension to the right or to the left, or by separating off other additional metrics, i.e. by passing from (1) to the decomposition (2) (see Lemma 1).

6. A semi-reducible decomposition (2) of the metric of a semi-reducible space V_n that is not reducible or $V(K)$ will be called **minimal** if its principal part ds_0^2 has the smallest dimension among all other semi-reducible decompositions of the metric ds^2. Hence, in particular, it follows that in a minimal decomposition (2) all additional metrics that can be separated off from its principal part have been separated off.

Theorem 3. The minimal semi-reducible decomposition (2) is uniquely determined*. Every other semi-reducible decomposition of the metric ds^2 is obtained from the minimal one by extension to the right by means of a semi-reducible splitting of some additional metric ds_ν^2.

The proof is by induction. For $n = 2$ the theorem is evidently satisfied, since a semi-reducible two-dimensional metric is represented in the form

$$ds^2 = du^2 + \sigma(u)\, dv^2$$

in a unique way, provided only that it does not have constant curvature, and this case is excluded. Suppose that the theorem is true for all V_q with $q < n$, and prove its validity also for V_n. To this end let us consider in (2) the system of equations (5), (6), and show that for every tensor A_{ij} satisfying it, the condition $A_{aa_\nu} = 0$ $(\nu = 1, \dots, p)$ is fulfilled. The latter is equivalent to $A_{u\lambda} = 0$ with respect to the decomposition

$$ds^2 = dt_0^2(x^\mu) + \sigma(x^\mu)\, dt_1^2(x^\lambda),$$

where dt_1^2 is an arbitrary metric among ds_ν^2 $(\nu = 1, \dots, p)$ in (2), and all the remaining terms are combined in dt_0^2. The required equality follows from Lemma

2 if dt_0^2 possesses one of the properties listed there. In the contrary case, dt_0^2 is a semi-reducible metric (with a non-one-dimensional principal part) of a space V_q that is not reducible or $V(K)$. Since $q < n$, by the induction hypothesis the minimal decomposition in dt_0^2 is uniquely determined. Using this fact, it is already not difficult to show that $A_{u\lambda} = 0$.

Thus, for any tensor A_{ij} satisfying (5), with respect to the minimal decomposition (2) the equality $A_{aa_\nu} = 0$ is fulfilled. This means that the tensor A_{ij} determines a semi-reducible decomposition either

* Up to, of course, trivial transformations that do not change the foliation of V_n into the coordinate hypersurfaces in (2).

coinciding with the given one, or obtained from it by extension to the right (Lemma 1). (Extension to the left is impossible owing to the minimality of the decomposition (2).) Since, under extension to the right, the dimension of the principal part increases, it follows that there can be only one minimal decomposition, and the theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.