MATHEMATICS

1962

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Abstract

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MATHEMATICS

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ASYMPTOTIC REPRESENTATIONS OF GENERATING FUNCTIONS AND LIMIT THEOREMS IN BOUNDARY-VALUE PROBLEMS

(Presented by Academician A. N. Kolmogorov on 8 XII 1961)

The finding of exact formulas for generating functions for the distribution of the time of first passage in a homogeneous random walk, for the distribution of the maximum of sums, etc., apparently is possible only in special cases (see, for example, ([2])). However, even under very general assumptions on the distribution function of a summand, close to the necessary ones, it is possible to isolate the principal parts of these functions—to obtain asymptotic representations that make it possible subsequently to carry out a fairly complete asymptotic analysis of the distributions themselves.

We shall consider here a sequence of independent identically distributed random variables $\xi_1, \xi_2, \ldots$, satisfying the conditions:

I. $\lambda_+ - \lambda_- > 0$, where

$$\lambda_- = \inf\{\lambda : \varphi(\lambda) < \infty\}, \qquad \lambda_+ = \sup\{\lambda : \varphi(\lambda) < \infty\},$$

$$\varphi(\lambda) = \mathbf{M}e^{-\lambda\xi_k}.$$

II. The distribution function has a nonzero absolutely continuous component.

We have $\varphi''(\lambda) > 0$ throughout the entire interval (λ_-, λ_+). Therefore there exist at most two real zeros $\lambda_\pm(z)$ $(\lambda_+(z) \geq \lambda_-(z))$ of the function $1 - z\varphi(\lambda)$, defined respectively for $z \in (z_\pm, z_0)$, where $z_\pm = \varphi^{-1}(\lambda_\pm)$,

$$z_0 = \frac{1}{\inf_{(\lambda_-,\lambda_+)} \varphi(\lambda)}.$$

Furthermore, the interval $[\lambda_-, \lambda_+]$ always contains a point λ_0 at which

$$\varphi(\lambda_0) = \inf_{(\lambda_-,\lambda_+)} \varphi(\lambda).$$

If $\lambda_- < \lambda_0 < \lambda_+$, the functions $\lambda_\pm(z)$ can be analytically continued to a neighborhood of the point z_0, which is their common branch point, at which they form a single circular system. Denote by $\mathcal{E}_\delta$ the domain obtained from the disk $|z| \le z_0 + \delta$ by deleting the points of the segment $[z_0, z_0+\delta]$; by $\mathcal{K}_\delta^\pm$ the domains

$$\{|\operatorname{Im} z| < \delta, \quad \operatorname{Re} z \ge 0, \quad z_\pm + \delta_1 \le |z| \le z_0 + \delta\}, \tag{1}$$

where the numbers δ and $\delta_1 = \delta_1(\delta)$ are chosen so that $\mathcal{K}_\delta^\pm$ contain no other singularities of the functions $\lambda_\pm(z)$, apart from the point z_0. Denote by $\tilde\lambda_\pm(z)$ the functions which coincide with $\lambda_\pm(z)$ at all points of the intervals $[z_\pm, z_0]$ and are equal to λ_0 for $z > z_0$.

Lemma. Under conditions I, II the function

$$W_z(\lambda) = \frac{1 - z\varphi(\lambda)}{(\lambda - \lambda_-(z))(\lambda - \lambda_+(z))}$$

for $z \in \mathcal{K}_\delta^+ \cap \mathcal{K}_\delta^-$ and sufficiently small $\delta > 0$, $\gamma > 0$ admits, uniquely up to factors independent of λ, the representation (factorization):

$$W_z(\lambda) = W_{z+}(\lambda) \cdot W_{z-}(\lambda) \qquad (\tilde\lambda_-(|z|) - \gamma \le \operatorname{Re}\lambda \le \tilde\lambda_+(|z|) + \gamma).$$

where the functions $W_{z\pm}(\lambda)$ are representable in the form

$$W_{z\pm}(\lambda) = \int_0^\infty e^{\mp\lambda t} w_{z\pm}(t)\, dt, \qquad e^{\mp(\tilde\lambda \mp (|z|) \mp \gamma)t} w_{z\pm}(t) \in L_1,$$

are defined and different from zero (at finite points), respectively, in the domains

$$z \in \mathcal{K}_{\delta_-}, \quad \operatorname{Re}\tilde\lambda \geqslant \lambda_-(|z|) - \gamma; \qquad z \in \mathcal{K}_{\delta_+}, \quad \operatorname{Re}\lambda \leqslant \tilde\lambda_+(|z|) + \gamma.$$

In these domains the functions $W_{z\pm}(\lambda)$ may be chosen regular in the aggregate of the variables z and λ.

Put

$$\xi_0 = 0, \qquad s_n = \sum_{k=0}^n \xi_k, \qquad \bar s_n = \max_{0 \le k \le n} s_k, \qquad x \geqslant 0, \qquad y \geqslant 0,$$

$$P_x^n = \mathbf{P}(\bar s_n \geqslant x), \qquad {}_1P_{x,y}^n = \mathbf{P}(\bar s_n \geqslant x,\ s_n < x - y), \qquad {}_2P_{x,y}^n = \mathbf{P}(s_n \geqslant x - y),$$

$${}_3P_{x,y}^n = \mathbf{P}(s_n < x - y), \qquad {}_4P_{x,y}^n = \mathbf{P}(s_n < x,\ s_n \geqslant x - y),$$

$${}_5P_{x,y}^n = \mathbf{P}(\bar s_{n-1} < x,\ x \leqslant s_n < x + y)$$

and denote by $P_x(z)$, ${}_jP_{x,y}(z)$ $(j = 1,2,3,4,5)$ the corresponding generating functions,

$$T(\lambda,\mu) = \frac{e^{\lambda x - \mu y}}{(\lambda^2 - \lambda_+(z)\lambda_-(z))W_{z+}(\lambda)W_{z-}(\mu)}.$$

The question of asymptotic representations of the generating functions is solved by the following theorems.

Theorem 1. As $x \to \infty,\ y \to \infty$

$$P_x(z) = T(\lambda_-(z),0) + (1-z)^{-1}O\Big(e^{x(\tilde\lambda_-(|z|)-\gamma)}\Big) \qquad (x>0,\ z \in \mathcal{E}_\delta \cap \mathcal{K}_{\delta_-}),$$

$${}_1P_{x,y}(z) = T(\lambda_-(z),\lambda_+(z)) + \lambda_+^{-1}(z)\begin{cases} O\Big(e^{x\lambda_-(z) - y(\tilde\lambda_+(|z|)+\gamma)}\Big) & (x>y,\ z \in \mathcal{E}_\delta \cap \mathcal{K}_{\delta_-} \cap \mathcal{K}_{\delta_+}),\\ O\Big(e^{x(\tilde\lambda_-(|z|)-\gamma) - y\lambda_+(z)}\Big) & (x<y,\ z \in \mathcal{E}_\delta \cap \mathcal{K}_{\delta_-} \cap \mathcal{K}_{\delta_+}),\end{cases}$$

$${}_2P_{x,y}(z) = T(\lambda_-(z),\lambda_-(z)) + O\Big(e^{(x-y)(\tilde\lambda_-(|z|)-\gamma)}\Big) \qquad (x>y,\ z \in \mathcal{E}_\delta \cap \mathcal{K}_{\delta_-}),$$

$${}_3P_{x,y}(z) = T(\lambda_+(z),\lambda_+(z)) + O\Big(e^{(x-y)(\tilde\lambda_+(|z|)+\gamma)}\Big) \qquad (x<y,\ z \in \mathcal{E}_\delta \cap \mathcal{K}_{\delta_+}).$$

Here the numbers δ,γ satisfy the conditions of the lemma; the estimates are uniform in z. The absolute values of the functions $P_x(z)$, ${}_jP_{x,y}(z)$ $(j=1,2,3)$ on the circle $|z| = \text{const}$ outside the strip $|\operatorname{Im} z| < \delta$, $\operatorname{Re} z > 0$ do not exceed the absolute values of these functions on the intersection of the circle with the boundary of, respectively, one of the domains $\mathcal{K}_{\delta_-}, \mathcal{K}_{\delta_+}$.

If the functions $\lambda_\pm(z)$ and the domains occurring in the lemma and in Theorem 1 are defined in the proper way, then condition I and $\lim_{|t|\to\infty} |\varphi(it)| < 1$ will be necessary for the assertions of the lemma and Theorem 1, so that conditions I, II are essential.

In the study of the joint distribution of $\bar s_n$ and s_n, the case $y = \text{const}$ is also of interest.

Theorem 2. If $z \in \mathcal{E}_\delta \cap \mathcal{K}_{\delta_-}$ and $x \to \infty$,

$${}_4P_{x,y}(z) = \frac{e^{x\lambda_-(z)}}{W_{z+}(\lambda_-(z))}\int_0^y \psi_z(t)\,dt + \overline{{}_4P}_{x,y}(z),$$

$$ {}_5P_{x,y}(z) = \frac{e^{x\lambda_-(z)}}{W_{z+}(\lambda_-(z))} \int_0^y w_{z+}(t)\,dt + {}_5\overline{P}_{x,y}(z), $$

where ${}_j\overline{P}_{x,y}(z) = O\left(e^{\widetilde{\lambda}_-(|z|)-\gamma}\right)$ $(j = 4, 5)$ uniformly in z, and for $j = 5$ also in y. The function $\psi_z(t)$ is defined by the representation $(\operatorname{Re}\mu < \operatorname{Re}\lambda_-(z))$

$$ \int_0^\infty e^{\mu t}\psi_z(t)\,dt = -\frac{1}{(\mu - \lambda_-(z))(\mu - \lambda_+(z))W_{z_-}(\mu)}. $$

The values of the functions ${}_jP_{x,y}(z)$ $(j = 4, 5)$ outside the strip $|\operatorname{Im} z| < \delta,\ \operatorname{Re} z > 0$ are estimated in the same way as in Theorem 1.

If $\lambda_0 = \lambda_-$ or $\lambda_0 = \lambda_+$, then the assertions of the theorems remain valid if the domains $\mathcal{K}_{\delta_\pm}$ are replaced by the domains $\widetilde{\mathcal{K}}_{\delta_\pm}$, which are obtained from $\mathcal{K}_{\delta_\pm}$ if, instead of the last inequality in (1), one puts $z_\pm + \delta_1 \le |z| \le z_0 - \delta$.

With the aid of Theorem 2 it is not difficult to establish that the functions $w_{z_+}(t)$, $\psi_z(t)$ have a simple physical meaning: they turn out to be the densities, respectively, of the conditional (with respect to the velocity of attainment of the boundary x) limiting distribution of the amount of the first overshoot over the barrier x and of the limiting distribution of the quantity $s_n - x$ under the conditions $s_n < x$,

$$ \lim_{n\to\infty} \frac{x}{n} < M\xi_k > 0. $$

The formulations of the theorems are adapted to the application of the transfer method. The method of their proof consists in finding an explicit form of the double transforms of the distributions sought, studying and using the analytic properties of these transforms (isolated pole, etc.).

We also give here one result of asymptotic analysis, following from Theorem 1 and pertaining to the probability ${}_1P^n_{x,y}$ in the case $M\xi_k = 0,\ D\xi_k = 1,\ x = o(n),\ y = o(n),\ \lambda_- < 0,\ \lambda_+ > 0$.

Theorem 3. Put $X = \dfrac{x}{\sqrt{n}},\ Y = \dfrac{y}{\sqrt{n}},\ \Phi(t) = \dfrac{1}{\sqrt{2\pi}} \int_t^\infty e^{-u^2/2}\,du.$

Then

$$ \mathbf{P}\left(\overline{s}_n \ge X\sqrt{n},\ s_n < (X - Y)\sqrt{n}\right) = $$

$$= \begin{cases} \Phi(X+Y) + e^{-\frac{1}{2}(X+Y)^2} \sum_{j=1}^{\infty} n^{-j/2} \Pi_{3j-1}(X,Y) + O\left(e^{-\gamma(x+y)}\right), & \text{for } X+Y = o(n^{1/8}), \\ e^{nH(X,Y)} \left\{ \frac{1}{X+Y} \Xi\left(\frac{1}{X+Y}, \frac{X}{\sqrt{n}}, \frac{Y}{\sqrt{n}}\right) + \frac{1}{\sqrt{n}} \Xi\left(n^{-1}, \frac{X}{\sqrt{n}}, \frac{Y}{\sqrt{n}}\right) \right\}, & \text{for } X+Y \to \infty. \end{cases}$$

Here Π_{3j-1} is a polynomial in its arguments of degree $3j-1$. Its coefficients are determined by $j+2$ moments of the random variable ξ_k and by the parameters

$$\frac{\partial^{s+k}}{\partial z^s \, \partial \mu^k} \frac{W_{z_+}(\mu)}{W_{z_+}(0)} \bigg|_{z=1,\ \mu=0}$$

for $s \geq 0, \ k \geq 1$ such that $2s+k \leq j$;

$$\sum n^{-j/2} \Pi_{3j-1}$$

is to be understood as an asymptotic expansion. $\Xi(...)$ also denotes an asymptotic expansion in nonnegative powers of the arguments. The function $H(X,Y)$ is a convergent series in powers of

$$\frac{X}{\sqrt{n}}, \quad \frac{Y}{\sqrt{n}}:$$

$$H(X,Y) = -\frac{1}{2n}(X+Y)^2 + \frac{M\xi_k^3}{6n^{3/2}} \left(X^3 + X^2Y - XY^2 - Y^3\right) + \cdots.$$

The coefficient of the leading term in the asymptotic expansion in the case $X+Y \to \infty$ is equal to

$$\frac{1}{\sqrt{2\pi}}.$$

The derivatives of the function

$$\frac{W_{z_+}(\mu)}{W_{z_+}(0)},$$

like the function $w_{z_+}(t)$, can be given a physical meaning.

The value of $\lim_{n\to\infty} P(\bar{s}_n \geq X\sqrt{n},\ s_n < (X-Y)\sqrt{n}) = \Phi(X+Y)$ under broader conditions is not difficult to obtain by considering the corresponding probability for the Brownian motion process and using the results of (1).

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Received
6 XII 1961

References

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3. A. A. Borovkov, *Theory of Probability and Its Applications*, **6**, 375 (1961).

Note: Figure translations are in progress. See original paper for figures.