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Abstract

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ON EXTREMAL QUASICONFORMAL MAPPINGS IN SPACE

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In the theory of Q-quasiconformal mappings of plane domains, extremal mappings (with the least possible Q) have been well studied. In this note we consider, in a certain sense, extremal mappings of some spatial domains. The proofs are carried out by the method of moduli (see ([1])) in space.

Let us introduce the following notation. By D we shall denote the cylindrical domain of three-dimensional Euclidean space (x, y, z), $(x, y) \in G$, $0 \le z \le H$, where G is a bounded closed domain in the (x, y)-plane. If some smooth non-self-intersecting surface passes inside D and intersects its bases, then it divides D into two parts, denoted by D_1 and D_2.

By $\{C\}$ we denote the family of all rectifiable curves lying inside D and joining its bases; into the family $\{C_1\}$ we include the curves from $\{C\}$ that lie entirely inside D_1, and into $\{C_2\}$ the curves from $\{C\}$ that lie entirely inside D_2. $M\{C\}$ is the modulus of the family $\{C\}$. The cylindroids $(x, y) \in G$, $0 \le z \le f(x, y)$, and $(x, y) \in G$, $f(x, y) \le z \le H$ will be denoted respectively by $\overline{D}_1$ and $\overline{D}_2$. It is assumed that $f(x, y)$ is continuous and $0 < f(x, y) < H$ everywhere in the domain G. The family of all rectifiable curves lying inside $\overline{D}_1$ (respectively inside $\overline{D}_2$) and joining its bases will be denoted by $\{\overline{C}_1\}$ (respectively $\{\overline{C}_2\}$).

1. **Lemma 1.** *In the relation*

$$M\{C_1\} + M\{C_2\} \le M\{C\} \tag{1}$$

   equality is attained only for a cylindrical surface with generators parallel to the z-axis.

Indeed, otherwise one can find on our surface such a point P_0 that, for any curve from $\{C_1\}$ (or perhaps from $\{C_2\}$) entering a certain ε-neighborhood $V_\varepsilon(P_0)$ of the point P_0, the length of the part of it lying outside $V_\varepsilon(P_0)$ will be greater than $H + 2\varepsilon$. Then the metric

$$\rho_1(P) = \begin{cases} \dfrac{1}{H}, & \text{if } P \in D_1 - V_\varepsilon(P_0), \\ 0, & \text{if } P \in D_1 \cap V_\varepsilon(P_0), \end{cases}$$

will be admissible for $\{C_1\}$, while $\rho_2(P) = \dfrac{1}{H}$, $P \in D_2$, is admissible for $\{C_2\}$.

Therefore

$$M\{C_1\} + M\{C_2\} \le \iiint_{D_1} \rho_1^3\, dV + \iiint_{D_2} \rho_2^3\, dV < \iiint_{D} \frac{1}{H^3}\, dV = M\{C\}.$$

The assertion concerning equality is obvious.

Lemma 2. *In the relation*

$$\frac{1}{\sqrt{M\{\overline{C}_1\}}} + \frac{1}{\sqrt{M\{\overline{C}_2\}}} \le \frac{1}{\sqrt{M\{C\}}} \tag{2}$$

equality is attained only in the case $f(x, y) \equiv \text{const}$.

Dividing G into parts G_i $(i = 1, 2, \dots, n)$ and considering the moduli of families of curves lying in the resulting cylinders $(x, y) \in G_i$, $0 < z < f(x, y)$; $(x, y) \in G_i$, $f(x, y) < z < H$, by Theorem 4 of [1], with the subsequent passage to the limit, we obtain the estimates

$$\iint_G \frac{d\sigma}{f^2(x, y)} \leqslant M\{\overline{C}_1\}, \qquad \iint_G \frac{d\sigma}{[H - f(x, y)]^2} \leqslant M\{\overline{C}_2\}.$$

In conjunction with the known inequality from [2]

$$\left(\frac{1}{S} \iint_G f^{-2}(x, y)\, d\sigma\right)^{-1/2} < \frac{1}{S} \iint_G f(x, y)\, d\sigma,$$

where $S = \iint_G d\sigma$ and $f(x, y) \not\equiv \text{const}$, these estimates give the required result. The case $f(x, y) \equiv \text{const}$ is trivial.

2. It next turns out to be expedient to introduce the following definition, proposed by B. V. Shabat.

Definition. A homeomorphic mapping $P_* = f(P)$ of a domain D onto D^* is called (Q_1, Q_2)-**quasiconformal** if it has continuous partial derivatives in D, its Jacobian is everywhere positive, and at each point $P \in D$ its principal linear part transforms a sphere into an ellipsoid in which the ratio of the major axis to the minor axis is bounded above by the quantity Q_1, and the ratio of the intermediate axis to the minor axis by the quantity Q_2. The inverse of a (Q_1, Q_2)-quasiconformal mapping is, obviously, also (Q_1, Q_2)-quasiconformal with the same Q_1 and Q_2.

Theorem 1. If $\{C^*\}$ is the image of the family of curves $\{C\}$ under a (Q_1, Q_2)-quasiconformal mapping, then

$$\frac{M\{C\}}{Q_1 Q_2} \leqslant M\{C^*\} \leqslant Q_1 Q_2 M\{C\}. \tag{3}$$

The proof is almost the same as that of Theorem 1 from [1].

3. Alongside D, consider in the space (u, v, w) the cylinder D^*, $(u, v) \in G^*$, $0 < w < h$. We shall assume that D^* can be obtained from D by the affine transformation

$$u = \lambda_1 x, \qquad v = \lambda_2 y, \qquad w = \frac{h}{H} z, \tag{4}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant h/H$. The notation $\{C^*\}$, $\{C_1^*\}$, D_1^*, etc., plays for D^* the same role as the corresponding notation without asterisks for D.

Let $\mathfrak{M}$ denote the class of (Q_1, Q_2)-quasiconformal mappings of D onto D^* that take the bases of D onto the bases of D^*.

Theorem 2. A mapping from the class $\mathfrak{M}$ possessing the minimal value of the product $Q_1 Q_2$ in comparison with the other mappings of the same class must necessarily have the form

$$u = u(x, y), \qquad v = v(x, y), \qquad w = \frac{h}{H} z. \tag{5}$$

Let Σ^* be an arbitrary cylindrical surface with generator along the w-axis, separating D^* into D_1^* and D_2^*. If one assumes that its preimage Σ under a mapping from the class $\mathfrak{M}$ is not a cylindrical surface with generator along the z-axis, then, by Lemma 1, taking into account inequalities (3), we obtain for such a mapping

$$M\{C\} > M\{C_1\} + M\{C_2\} \geqslant \frac{M\{C_1^*\}}{Q_1 Q_2} + \frac{M\{C_2^*\}}{Q_1 Q_2} = \frac{M\{C^*\}}{Q_1 Q_2},$$

or

$$Q_1 Q_2 > \frac{M\{C^*\}}{M\{C\}}.$$

The mapping (4) belongs to the class $\mathfrak{M}$, and for it

$$Q_1 Q_2 = \frac{M\{C^*\}}{M\{C\}}.$$

Consequently, for the extremal mapping (minimizing $Q_1 Q_2$), Σ is a cylindrical surface with generators along the z-axis. Hence it is easy to obtain that any segment parallel to the z-axis is transformed under the extremal mapping into a segment parallel to the w-axis.

Thus, for the extremal mapping, the image of the plane $z = z_0$ can only be a surface $w = w_0(u, v)$.

Suppose that $w_0(u,v) \not\equiv \text{const}$. Then Lemma 2 and inequalities (3) give

$$\frac{1}{\sqrt{M\{C\}}} = \frac{1}{\sqrt{M\{C_1\}}} + \frac{1}{\sqrt{M\{C_2\}}} \le \frac{\sqrt{Q_1 Q_2}}{\sqrt{M\{C_2^*\}}} + \frac{\sqrt{Q_1 Q_2}}{\sqrt{M\{C_2^*\}}} < \sqrt{Q_1 Q_2}\,\frac{1}{\sqrt{M\{C^*\}}},$$

whence

$$Q_1 Q_2 > \frac{M\{C^*\}}{M\{C\}},$$

which contradicts the extremality of the mapping.

The linearity

$$w = \frac{h}{H}\,z$$

is easily proved by contradiction.

Theorem 3. *The extremal mapping has constant Jacobian.*

This assertion follows easily from the relations:

$$S^* h = \iiint_D J\,dV = \frac{h}{H}\iiint_D \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\,dV \le \frac{h^3}{H^3}\,SHQ_1Q_2,$$

where

$$S^* = \iint_{G^*} d\sigma, \qquad J = \frac{D(u,v,w)}{D(x,y,z)}.$$

Remark. For the extremal mapping (5), the corresponding plane mapping of G onto G^*,

$$u = u(x,y), \qquad v = v(x,y) \tag{6}$$

will be Q-quasiconformal, with

$$Q = \frac{Q_1^2}{J}\,\frac{h^3}{H^3}.$$

Let us apply to the mapping (6) the theorem on the extremal quasiconformal mapping in the plane (3).

Theorem 4. *Among all extremal mappings of the parallelepiped* $0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq H$ *onto the parallelepiped* $0 \leq u \leq \lambda_1 a,\ 0 \leq v \leq \lambda_2 b,\ 0 \leq w \leq h$ *with corresponding edges, the affine mapping is the unique one having the minimal value of* Q_1.

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Note: Figure translations are in progress. See original paper for figures.