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Abstract

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MATHEMATICS

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ON THE BEHAVIOR OF THE SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM WITH ZERO BOUNDARY CONDITIONS FOR A GENERAL PARABOLIC EQUATION

(Presented by Academician I. G. Petrovskii, 15 XI 1961)

0. Questions concerning the behavior as $t \to +\infty$ of the solution of the first boundary-value problem and the Cauchy problem for a general parabolic equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = 0 \qquad (1)$$

were considered in the works (^{2,3}). In the present note we investigate the question of the character of the decrease as $t \to +\infty$ of the solution of the first boundary-value problem with zero boundary conditions for equation (1), depending on the domain.

Consider the heat-conduction equation

$$a_0 \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0.$$

The solution of the first boundary-value problem with zero boundary conditions for this equation in the domain $G = \{-\varepsilon < x_1 < \varepsilon, \dots, -\varepsilon < x_n < \varepsilon;\ t > 0\}$ has the form

$$u(x,t) = w(x,t)\exp\left[-\frac{a_0 n\pi^2}{4}\frac{t}{\varepsilon^2}\right] = w(x,t)\exp\left[-\frac{a_0 n\pi^2}{4}\int_0^t \frac{d\tau}{\varepsilon^2}\right],$$

where $w(x,t)$ is a bounded function depending on the initial function, a_0, and ε.

We shall show that the solution of equation (1) behaves in a similar way.

1. Let G_1 be a domain situated in $\Pi_{\psi(t)} = \{-\psi(t) < x_1 < \psi(t);\ t > 0\}$ of the space of variables $(x, t) = (x_1, \dots, x_n; t)$ and having boundary points on the plane $t = 0$. Denote by B_h the intersection $\overline{G}_1 \cap \{t = h\}$ for any $h \geq 0$, and by $\dot{G}_1$ the lateral boundary of the domain G_1. Obviously, $\overline{G}_1 \setminus G_1 = \dot{G}_1 \cup B_0$. We shall assume that B_h is bounded for every finite $h \geq 0$.

Consider equation (1) with coefficients defined in G_1 and satisfying there the following conditions: a) $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t) \in C^{(0)}$; b)

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0 \sum_{i=1}^{n} \xi_i^2$$

for any point $(x,t) \in G_1$ and for any real vector $(\xi_1, \dots, \xi_n) \neq 0$; c) $b_1(x,t) \leq B/2$; d) $c(x,t) \leq 0$ (a_0, B are positive constants).

By a solution $u(x,t)$ of equation (1) we shall understand a function continuous in $\overline{G}_1$, having in G_1 a first continuous derivative with respect to t and second continuous derivatives with respect to x_i and x_j.

Theorem 1. Let the function $x_1 = \psi(t)$ satisfy, for $t \geq 0$, the requirements: 1) $\psi(t) \in C^{(1)}$; 2) $0 < \psi(t) \leq 1$; 3) $-B/16H \leq \psi'(t) \leq 0$, ($H = e^{B/a_0}$).

Let $u(x,t)$ be a solution of equation (1) in G_1, for which the condition

$$u(x,t) = 0 \quad \text{for } (x,t) \in \dot{G}_1 \tag{2}$$

is satisfied. Then in $\overline{G}_1$

$$|u(x,t)| < 2 \max_{(x,0)\in B_0} |u(x,0)| \exp\left[-\frac{a_0}{8H}\int_0^t \frac{d\tau}{\psi^2(\tau)}\right].$$

For the proof, consider the particular solution

$$z(x_1,t) = -\frac{a_0}{B^2}\exp\left[\frac{B}{2a_0}(\psi(t) - x_1)\right] - \frac{x_1}{2B} - \frac{\psi(t)}{2B} +$$

$$+\frac{1}{2a_0}\psi^2(t)\exp\left[\frac{B}{a_0}\psi(t)\right] + \frac{a_0}{B^2}\exp\left[\frac{B}{a_0}\psi(t)\right]$$

of the equation

$$a_0\frac{\partial^2 z}{\partial x_1^2} + \frac{B}{2}\frac{\partial z}{\partial x_1} + \frac{1}{4} = 0,$$

satisfying the conditions

$$\frac{\partial z(x_1,t)}{\partial x_1} \geq 0, \qquad \frac{\partial^2 z(x_1,t)}{\partial x_1^2} \leq 0,$$

$$0 < \frac{1}{2a_0}\psi^2(t)\exp\left[\frac{B}{a_0}\psi(t)\right] = z(-\psi(t),t) \leq z(x_1,t) \leq z(\psi(t),t) <$$

$$< \frac{1}{a_0}\psi^2(t)\exp\left[\frac{B}{a_0}\psi(t)\right] \quad \text{for } -\psi(t) \leq x_1 \leq \psi(t),\ t \geq 0.$$

Put $V(x,t) = z(x_1,t)e^{-\lambda(t)}$. Since

$$\frac{\partial V}{\partial x_i} = 0, \qquad \frac{\partial^2 V}{\partial x_i^2} = 0, \qquad \frac{\partial^2 V}{\partial x_i \partial x_k} = 0 \quad (i = 2,\dots,n;\ k = 1,2,\dots,n),$$

we have

$$e^{\lambda(t)}\mathcal{L}V \equiv a_{11}(x,t)\frac{\partial^2 z}{\partial x_1^2} + b_1(x,t)\frac{\partial z}{\partial x_1} + c(x,t)z - \frac{\partial z}{\partial t} + z\lambda'(t).$$

The rest of the proof repeats verbatim the proof of Theorem 2 of paper (4).

2. Consider the special case of the domain G_1—the domain

$$G_2 = \{\,0 \leq r < \psi(t);\ t > 0\,\} \quad \left(r = \sqrt{x_1^2 + \dots + x_n^2}\right).$$

Let the coefficients of equation (1) in G_2 satisfy the requirements:

a′) $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t) \in C^{(0)}$;

b′)

$$A\sum_{i=1}^{n}\xi_i^2 \geq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0\sum_{i=1}^{n}\xi_i^2$$

for any point $(x,t) \in G_2$ and any real vector $(\xi_1,\dots,\xi_n) \neq 0$;

c′)

$$B \geq \left[\sum_{i=1}^{n} b_i^2(x,t)\right]^{1/2} \geq 0;$$

d′)

$$0 \geq c(x,t) \geq -c_0,$$

where a_0, A, B, C_0 are positive constants.

Lemma. Let the function $\psi(t)$ satisfy, for $t \geq 0$, the conditions:

1′) $\psi'(t) \in C^{(1)}$;
2′) $0 < \psi(t) \leq \Psi$;
3′) $|\psi'(t)| \leq N$,

where N, Ψ are positive constants.

Let $u(x,t)$ be a solution of equation (1) in G_2, satisfying the condition

$$u(x,t) \geq 0 \quad \text{on } \overline{G}_1 \setminus G_1,$$

$$u(x_0,0) > 0 \quad \text{for } (x_0,0) \in B_0. \tag{3}$$

Then for any $t_1 > 0$ one can find such a $k_1 > 0$ that, for $0 \leq r \leq \psi(t_1)$,

$$u(x,t_1) \geq k_1 \cos^2 \frac{\pi r^2}{2\psi^2(t_1)},$$

where $k_1 > 0$ depends on $n, a_0, A, B, c_0, \Psi, N, t_1, \min_{[0,2t_1]} \psi(t)$.

In the proof of the lemma, Theorem 6 of the work (^{1}) is used.

Theorem 2. Let the function $\psi(t)$, for $t \geq 0$, satisfy conditions 1′)–3′) of the lemma. Let $u(x,t)$ be a solution of equation (1) in G_2, satisfying conditions (3). Then in

$$G_2^{t_1} = \{0 \leq r \leq \psi(t),\ t \leq t_1\}$$

$$u(x,t) \geq k_2 \cos^2 \frac{\pi r^2}{2\psi^2(t)} \exp\left[-\gamma \int_0^t \frac{d\tau}{\psi^2(\tau)}\right],$$

where

$$k_2 = k_1 \exp\left[\gamma \int_0^{t_1} \frac{d\tau}{\psi^2(\tau)}\right],$$

γ is a positive constant depending on $n, a_0, A, B, c_0, \Psi, N$.

Proof. Consider in $\overline{G}_2$ the function

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$$v(x,t) = y(x,t)e^{-\lambda(t)} = \cos^2 \frac{\pi r^2}{2\psi^2(t)} e^{-\lambda(t)},$$

where $\lambda(t) > 0,\ \lambda'(t) > 0$. For brevity put $\xi = \pi r^2/2\psi^2(t)$. We have in G_2

$$\begin{aligned} e^{\lambda(t)}\mathcal{L}v &= \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial y}{\partial x_i} + c(x,t)y - \frac{\partial y}{\partial t} + y\lambda'(t) \\ &= \frac{2\pi^2}{\psi^4(t)}\sin^2\xi \sum_{i,j=1}^{n} a_{ij}(x,t)x_i x_j - \frac{2\pi^2}{\psi^4(t)}\cos^2\xi \sum_{i,j=1}^{n} a_{ij}(x,t)x_i x_j \\ &\quad - \frac{2\pi}{\psi^2(t)}\sin\xi\cos\xi \sum_{i=1}^{n} a_{ii}(x,t) - 2\cos\xi\sin\xi\,\frac{\pi}{\psi^2(t)}\sum_{i=1}^{n} b_i(x,t)x_i \\ &\quad + c(x,t)\cos^2\xi - 2\cos\xi\sin\xi\,\frac{\pi r^2\psi'(t)}{\psi^3(t)} + \cos^2\xi\,\lambda'(t). \end{aligned}$$

For $\sqrt{1-\alpha}\,\psi(t) \le r < \psi(t)$ we have, for any $\lambda(t) > 0$ for which $\lambda'(t) > 0$,

$$\begin{aligned} e^{\lambda(t)}\mathcal{L}v &\ge \frac{2\pi^2}{\psi^2(t)}a_0(1-\alpha)\cos^2\alpha\frac{\pi}{2} - \frac{2\pi^2}{\psi^2(t)}A\sin^2\alpha\frac{\pi}{2} - \frac{2\pi nA}{\psi^2(t)}\sin\alpha\frac{\pi}{2} \\ &\quad - \frac{2\pi B}{\psi(t)}\sin\alpha\frac{\pi}{2} - c_0\sin^2\alpha\frac{\pi}{2} - \frac{2\pi N}{\psi(t)}\sin\alpha\frac{\pi}{2} \\ &\ge \frac{1}{\psi^2(t)}\Big[2\pi^2 a_0(1-\alpha)\cos^2\alpha\frac{\pi}{2} - 2\pi^2 A\sin^2\alpha\frac{\pi}{2} - 2\pi nA\sin\alpha\frac{\pi}{2} \\ &\quad -2\pi B\Psi\sin\alpha\frac{\pi}{2} - c_0\Psi^2\sin^2\alpha\frac{\pi}{2} - 2\pi N\Psi\sin\alpha\frac{\pi}{2}\Big] = \frac{R}{\psi^2(t)}. \end{aligned}$$

If $\alpha > 0$ is sufficiently small, then $R > 0$; the constant α depends on $a_0, A, B, c_0, \Psi, N, n$.

Put

$$\beta = \sin^2\frac{\alpha\pi}{2}.$$

For $0 \le r \le \sqrt{1-\alpha}\,\psi(t)$ we have

$$\begin{aligned} e^{\lambda(t)}\mathcal{L}v &\ge -\frac{2\pi^2 A}{\psi^2(t)} - \frac{\pi^2 nA}{\psi^2(t)} - \frac{B\pi^2}{\psi(t)} - c_0 - \frac{\pi^2 N}{\psi(t)} + \beta\lambda'(t) \\ &> \frac{1}{\psi^2(t)}\left[-\pi^2 A(n+2) - Bn^2\Psi - c_0\Psi^2 - \pi^2 N\Psi + \psi^2\beta\lambda'(t)\right] = \frac{T}{\psi^2(t)}. \end{aligned}$$

Set

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$$\lambda(t) = \frac{\omega}{\beta} \int_{t_1}^{t} \frac{d\tau}{\psi^2(\tau)},$$

then

$$\lambda'(t) = \frac{\omega}{\beta} \frac{1}{\psi^2(t)},$$

$$T = -\pi^2 A(n+2) - B\pi^2\Psi - c_0\Psi - \pi^2 N\Psi + \omega > 0,$$

if $\omega > 0$ is sufficiently large. By the lemma, on B_{t_1}

$$u(x,t) \geq k_1 \cos^2 \frac{\pi r^2}{2\psi^2(t_1)},$$

therefore, on $\overline{G}_2^{t_1} \setminus G_2^{t_1}$,

$$u(x,t) \geq k_1 \cos^2 \frac{\pi r^2}{2\psi^2(t)} \exp\left[-\gamma \int_{t_1}^{t} \frac{d\tau}{\psi^2(\tau)}\right] = k_1 v(x,t) \left(\gamma = \frac{\omega}{\beta} > 0\right). \quad (4)$$

Since $\mathcal{L}v > 0$ in $G_2^{t_1}$, inequality (4) is valid in $\overline{G}_2^{t_1}$. The theorem is proved.

Clearly, Theorems 1 and 2 are valid for quasilinear parabolic equations.

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REFERENCES

1. L. Nirenberg, Comm. Pure and Appl. Math., 6, 167 (1953).
2. A. Friedman, J. Math. and Mech., 8, No. 1, 57 (1959).
3. A. M. Il’ in, Uspekhi Mat. Nauk, 16, issue 2 (98), 115 (1961).
4. Yu. N. Cheremnykh, Izv. AN SSSR, ser. matem., 23, 913 (1959).

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