MATHEMATICS

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Abstract

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LIMITING BEHAVIOR OF COMPOSITIONS OF MEASURES IN SOME SYMMETRIC SPACES

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1. Consider a symmetric space M on which a group of motions G acts. Denote by K the stationary subgroup of some point $e \in M$. We shall call a probability measure μ, defined on all Borel subsets of M, **symmetric** if for any Borel set $\Gamma \subset M$ and any $k \in K$ the equality $\mu(\Gamma) = \mu(k\Gamma)$ holds. With a sequence of symmetric measures $\mu_1, \mu_2, \dots, \mu_m, \dots$ there is associated a Markov chain $x(m)$, $m = 0, 1, \dots$, invariant with respect to motions, for which the probability $P^m_{m+1}(x, \Gamma)$ of passing during the time $(m, m+1)$ from the point x into the set Γ is equal to $\mu_{m+1}(g_x^{-1}\Gamma)$, where g_x is any motion carrying e into x. The transition probabilities for the time $(0, m)$ are determined by the recurrent formula:

$$P^0_m(x, \Gamma) = \int_M P^0_{m-1}(x, dy)\, P^{m-1}_m(y, \Gamma).$$

We shall call the measure $P^0_m(e, \Gamma)$ the **composition** $\mu_1 * \dots * \mu_m$ of the measures μ_i, $i = 1, \dots, m$. In the case when $\mu_1 = \dots = \mu_m = \mu$, we shall write $\mu_1 * \dots * \mu_m = \mu^m$. We shall consider the question of the behavior of the measure μ^m as $m \to \infty$. The results are formulated for the case when G is the complex unimodular group. Analogous results are valid for other classical complex groups.

2. Let M be the set of positive-definite Hermitian unimodular matrices of order n; G the group of unimodular matrices acting on M according to the formula $g : x \to gxg^-$ $(x \in M)$. The stationary subgroup of the identity matrix e is the unitary group K.

Denote by D the set of points $t = (t_1, \dots, t_n)$ of Euclidean space defined by the equality

$$D = \{t : t_1 + \dots + t_n = 0,\ t_1 \geq t_2 \geq \dots \geq t_n\}.$$

To each matrix $x \in M$ we associate the collection $t(x)$ of logarithms of its eigenvalues, arranged in decreasing order. Obviously, $t(x) \in D$ for any $x \in M$. Every measure μ on M induces a measure $\bar{\mu}$ on D by the formula $\bar{\mu}(A) = \mu\{x : t(x) \in A\}$. Moreover, an arbitrary measure on D is induced by one and only one symmetric measure on M. In what follows we shall denote a symmetric measure on M and the measure induced by it on D by the same letter.

3. Let $\Phi(\rho, x)$ be a bounded zonal function on M ($\rho = (\rho_1, \dots, \rho_n)$ is a complex vector), i.e., an eigenfunction of the Laplace operators on M, invariant with respect to the stationary subgroup K ([1]). The function $\Phi(\rho, x)$ is uniquely determined by the set of eigenvalues (which are expressed in terms of ρ) and by the condition $\Phi(\rho, e) = 1$. We shall call the function

$$f_\mu(\rho) = \int_M \Phi(\rho, x)\, \mu(dx) = \int_D \Phi(\rho, t)\, \mu(dt),$$

where $\Phi(\rho, t)$ is the value of the function $\Phi(\rho, x)$ on matrices $x \in M$ such that $t(x) = t$, the **characteristic function** of the measure μ. With the invariant Markov chain $x(m)$ (see item 1) there are associated the operators T_m^0, defined by the formula

$$T_m^0 \varphi(x) = \int_M \varphi(y)\, P_m^0(x, dy),$$

permutable-

···with all shifts. It can be shown that

$$T_m^0 \Phi(\rho, x) = f_{\mu_1}(\rho) \dots f_{\mu_m}(\rho) \Phi(\rho, x).$$

Putting $x = e$, it follows that

$$f_{\mu_1 * \dots * \mu_m}(\rho) = \prod_{i=1}^{m} f_{\mu_i}(\rho).$$

For the case of the unimodular group G considered by us, the functions $\Phi(\rho, t)$ have the form (see (2)):

$$\Phi(\rho, t) = \frac{c \det \| e^{\rho_i t_j} \|}{\prod_{i<j} (\rho_i - \rho_j) \operatorname{sh} \dfrac{t_i - t_j}{2}}, \qquad i, j = 1, \dots, n.$$

4. Lemma. Let $\rho = \sigma + i\tau$, where

$$\sigma = \left\{ \frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2} \right\},$$

and τ is a real vector. The functions $\Phi(\rho, t) = \Phi(\sigma + i\tau, t)$ satisfy the following conditions:

1°. $\Phi(\sigma+i\tau,t)\big|_{\tau=0}=1$ for any $t\in D$; for any τ, $\Phi(\sigma+i\tau,t)$ is bounded as a function of $t\in D$.

2°. For any t, $\Phi(\sigma+i\tau,t)$ is a positive-definite function of τ.

3°. The inverse Fourier transform (in τ) $\widetilde{\Phi}_t$ of the function $\Phi(\sigma+i\tau,t)$ (which, by 1° and 2°, is a probability measure in Euclidean space) satisfies the condition: for any $\varepsilon>0$,

$$\lim_{m\to\infty}\mu^m\left\{t:\lim_{a\to\infty}\widetilde{\Phi}_t\{y:\|y-t\|>a\}>\varepsilon\right\}=0$$

for any symmetric measure μ on M (here $\|y\|=\max_{1\le i\le n}|y_i|$).

4°. For no fixed $t\neq 0$ does the equality

$$\Phi(\sigma+i\tau,t)=e^{i\tau_k a_k}\varphi(\tau_1,\dots,\tau_{k-1},\tau_{k+1},\dots,\tau_n)$$

hold identically in τ, where φ does not depend on τ_k.

From conditions 1° and 2° it follows that, for any measure μ on M, the function (of τ $f_\mu(\sigma+i\tau)$ is the characteristic function (in the usual sense) of some probability measure $\tilde{\mu}$ in Euclidean space. Moreover,

$$\widetilde{\mu^m}=(\tilde{\mu})^m$$

(where the composition on the right is understood in the usual sense).

Denote by $\hat{\Gamma}_c$, where $c=(c_1,\dots,c_n)$, $-\infty<c_i<\infty$, the set $\{t:t_i<c_i,\ i=1,\dots,n\}$ in Euclidean space, and by Γ_c the set $\hat{\Gamma}_c\cap D$.

Theorem. As $m\to\infty$,

$$\sup_c\left|\mu^m(\Gamma_c)-\tilde{\mu}^m(\hat{\Gamma}_c)\right|\to 0.$$

Proof. There is the formula

$$\tilde{\mu}^m(\hat{\Gamma}_c)=\int_D\widetilde{\Phi}_t(\hat{\Gamma}_c)\,\mu^m(dt),$$

from which it follows that, under conditions 1°–3° of the lemma, for any $\varepsilon>0$ and $a>a(\varepsilon)$,

$$\tilde{\mu}^m(\hat{\Gamma}_{c-a})\le\mu^m(\Gamma_c)+\varepsilon+\delta'_m,$$

$$\tilde{\mu}^m(\hat{\Gamma}_{c+a}) \geq \mu^m(\Gamma_c) - \varepsilon + \delta''_m,$$

where δ'_m and δ''_m tend to zero as $m \to \infty$, uniformly in c ($c \pm a = \{c_1 \pm a, \dots, c \pm a\}$). From condition 4° it follows that the measure $\tilde{\mu}$ does not concent···

is not concentrated on any plane of the form $t_k = a_k$. Hence it follows that, for any a,

$$\lim_{m\to\infty} \sup_c \tilde{\mu}^m\{t:\ c_k - a < t_k < c_k + a\} = 0.$$

Consequently,

$$\lim_{m\to\infty} \sup_c \left|\tilde{\mu}^m(\tilde{\Gamma}_c) - \tilde{\mu}^m(\Gamma_{c\pm a})\right| = 0.$$

The theorem is proved.

Remark. It is not hard to verify that for any measure μ the measure $\tilde{\mu}$ is concentrated on the plane $t_1 + \dots + t_n = 0$.

We shall say that the compositions of the measures μ_1 and μ_2 approach each other if

$$\lim_{m\to\infty} \sup_c |\mu_1^m(\Gamma_c) - \mu_2^m(\Gamma_c)| = 0.$$

Corollary 1. In order that the compositions of the measures μ_1 and μ_2 approach each other, it is necessary and sufficient that the compositions (in the ordinary sense) of the measures $\tilde{\mu}_1$ and $\tilde{\mu}_2$ approach each other, i.e., that

$$\lim_{m\to\infty} \sup_c \left|\tilde{\mu}_1^m(\tilde{\Gamma}_c) - \tilde{\mu}_2^m(\tilde{\Gamma}_c)\right| = 0.$$

Corollary 2. If the function $f_\mu(\sigma + i\tau)$ has continuous second partial derivatives with respect to τ_i, τ_k, then the probability distribution μ^m of the vector $(t_1(x), \dots, t_n(x))$ is asymptotically normal with parameters

$$\left\{ \frac{1}{i} \frac{\partial f_\mu}{\partial \tau_k}\bigg|_{\tau=0},\ k = 1, \dots, n \right\}, \qquad \left\{ \left[-\frac{\partial^2 f_\mu}{\partial \tau_k \partial \tau_l} + \frac{\partial f_\mu}{\partial \tau_k} \frac{\partial f_\mu}{\partial \tau_l} \right]_{\tau=0},\ k, l = 1, \dots, n \right\}.$$

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Note: Figure translations are in progress. See original paper for figures.