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Abstract

Full Text

Mathematics

M. Z. Solomyak

On Spaces Conjugate to the Spaces W_p^l of S. L. Sobolev

(Presented by Academician V. I. Smirnov on 3 XII 1961)

1. One of the possible approaches to the construction of the general form of a linear functional in the spaces $W_p^l(\Omega)$ of S. L. Sobolev ⁽¹⁾ was developed by Lax ^{(2)*}. Namely, every function $\varphi(x)$, continuous and bounded in the domain Ω , generates the functional

$$f(u) = \int_{\Omega} \overline{\varphi(x)} u(x) dx$$

with norm

$$\|f\| = \sup \frac{f(u)}{\|u\|_{W_p^l}}. \quad (1)$$

The closure in the metric (1) of the set of all continuous bounded functions then leads to the space $[W_p^l(\Omega)]^*$.

It is of interest, however, to give a more concrete description of the “generalized elements” of the space $[W_p^l(\Omega)]^*$. In the present note we show that in the case when Ω is a bounded domain with sufficiently smooth boundary and $p > 1$, the space $[W_p^l(\Omega)]^*$, as a set of elements, coincides with $W_q^l(\Omega)$, where $q = p/(p-1)$; the norm in $[W_p^l]^*$ is metrically equivalent, but for $p \neq 2$ is not identical, to the norm in W_q^l .

2. Let Ω be a bounded domain of n -dimensional Euclidean space with sufficiently smooth boundary Γ . In addition to the spaces $W_p^l(\Omega)$, we shall consider their subspaces $\overset{\circ}{W}_p^l(\Omega)$, which are the closure, in the metric of W_p^l , of the set of smooth functions equal to zero in a boundary strip. The norm in $\overset{\circ}{W}_p^l(\Omega)$ is defined, as usual, by the equality

$$\|u\|_{\overset{\circ}{W}_p^l(\Omega)} = \left\{ \int_{\Omega} \left(\sum_{|\alpha|=l} |D^\alpha u|^2 \right)^{p/2} dx \right\}^{1/p}; \quad (2)$$

it will be convenient for us to define the norm in $W_p^l(\Omega)$ as

$$\|u\|_{W_p^l(\Omega)} = \left\{ \int_{\Omega} \left(|u|^2 + \sum_{|\alpha|=l} |D^\alpha u|^2 \right)^{p/2} dx \right\}^{1/p}. \quad (3)$$

Theorem 1. Every linear functional in the space $\mathring{W}_p^l(\Omega)$ admits a unique representation of the form

$$f(u) = \int_{\Omega} \sum_{|\alpha|=l} \overline{D^\alpha v} D^\alpha u \, dx, \quad (4)$$

* The results of the present note are valid both for the real and for the complex space W_p^l ; it is more convenient for us to present the exposition for the complex case.

where $v \in \mathring{W}_q^l(\Omega)$; moreover

$$\|f\| \leq \|v\|_{\mathring{W}_q^l(\Omega)}. \quad (5)$$

Theorem 2. Every linear functional in the space $W_p^l(\Omega)$ admits a unique representation of the form

$$f(u) = \int_{\Omega} \left(\bar{v}u + \sum_{|\alpha|=l} \overline{D^\alpha v} D^\alpha u \right) dx, \quad (6)$$

where $v \in W_q^l(\Omega)$; moreover, the norm of the functional (6) does not exceed $\|v\|_{W_q^l}$.

Remark. Inequality (5) is obvious; by the well-known theorem of Banach (see, for example, (3)), it implies the equivalence of the norms in $(\mathring{W}_p^l)^*$ and \mathring{W}_q^l , and also in $(W_p^l)^*$ and W_q^l .

3. The question of finding the general form of a linear functional in the spaces W_p^l and \mathring{W}_p^l turns out to be closely connected with the question of the solvability of certain integral identities.

We denote: $\xi = (\xi_1, \xi_2, \dots, \xi_n)$; α is the set of indices $\alpha_1, \alpha_2, \dots, \alpha_n$; $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$; $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_n^{\alpha_n}$.

Let $c(x) > 0$ and

$$P(\xi_1, \xi_2, \dots, \xi_n) = \sum_{|\alpha|=|\beta|=l} a_{\alpha\beta}(x) \xi_\alpha \bar{\xi}_\beta$$

be a uniformly positive form in $\Omega \cup \Gamma$ of order $2l$ with coefficients that are continuously differentiable l times. Consider the integral identities

$$\int_{\Omega} \sum_{|\alpha|=|\beta|=l} a_{\alpha\beta}(x) D^\alpha v \cdot \overline{D^\beta \varphi} dx = \int_{\Omega} \sum_{|\alpha|=l} g_\alpha(x) \overline{D^\alpha \varphi} dx, \quad (7)$$

$$\int_{\Omega} \left[c(x) v \overline{\varphi} + \sum_{|\alpha|=|\beta|=l} a_{\alpha\beta}(x) D^\alpha v \cdot \overline{D^\beta \varphi} \right] dx = \int_{\Omega} \left[c(x) g(x) \overline{\varphi} + \sum_{|\alpha|=l} g_\alpha(x) \overline{D^\alpha \varphi} \right] dx, \quad (8)$$

where $g(x)$ and $g_\alpha(x)$ are given functions belonging to the space $L_q(\Omega)$, $q > 1$.

Theorem 3. There exists a unique function $v(x)$, belonging to the space $\dot{W}_q^l(\Omega)$, and satisfying the integral identity (7) for all $\varphi(x) \in \dot{W}_p^l(\Omega)$. For the norm of the solution $v(x)$ the inequality holds

$$\|v(x)\|_{\dot{W}_q^l(\Omega)} \leq C \sum_{|\alpha|=l} \|g_\alpha(x)\|_{L_q(\Omega)}. \quad (9)$$

Theorem 4. There exists a unique function $v(x)$, belonging to the space $W_q^l(\Omega)$, and satisfying the integral identity (8) for all $\varphi(x) \in W_p^l(\Omega)$. For the norm of the solution $v(x)$ the inequality holds

$$\|v(x)\|_{W_q^l(\Omega)} \leq C \left[\|g(x)\|_{L_q(\Omega)} + \sum_{|\alpha|=l} \|g_\alpha(x)\|_{L_q(\Omega)} \right]. \quad (10)$$

We note that the solutions of the integral identities (7) and (8) are weak solutions of certain boundary-value problems for elliptic equations. The proof of Theorems 3 and 4 is carried out by a method essentially ana-

logical to that which was used in the study of strong solutions by A. I. Koshelev (⁴) (see also the author's paper (⁵)). The central point in the proof is obtaining a priori estimates (9) or (10) for the solution of a boundary-value problem for an operator with constant coefficients in a half-space. In contrast to (⁴, ⁵), here it turns out to be more expedient not to pass to the differential equation, but to use directly the original integral identity in order to obtain the required estimates.

Let us illustrate this by the example of an estimate for the solution of the integral identity (8) in the half-space $x_n \geq 0$; the coefficients c and $a_{\alpha\beta}$ are assumed

constant, and the functions $v(x)$, $g(x)$, and $g_\alpha(x)$ are assumed finite. Let $\zeta(x)$ be a cut-off function identically equal to one on the union of the supports of the functions $v(x)$, $g(x)$, $g_\alpha(x)$; denote by $\lambda_j = \lambda_j(\xi_1, \xi_2, \dots, \xi_{n-1})$, $j = 1, 2, \dots, l$, those roots of the equation $P(\xi_1, \xi_2, \dots, \xi_{n-1}, \lambda) + c = 0$ for which $\text{Im } \lambda_j > 0$. Substituting successively in identity (8)

$$\varphi(x) = \zeta(x) \exp \left[-i \sum_{k=1}^n \xi_{kx} k \right] \quad \text{and} \quad \varphi(x) = \zeta(x) \exp \left[-i \left(\lambda_{jx} n + \sum_{k=1}^{n-1} \xi_{kx} k \right) \right],$$

$j = 1, 2, \dots, l$, we obtain relations from which the Fourier transforms of the function $v(x)$ and of its derivatives are easily expressed in terms of the Fourier transforms of the functions $g(x)$ and $g_\alpha(x)$. After this, S. G. Mikhlin's theorem (6) on multipliers of Fourier integrals is used.

4. We shall now show, restricting ourselves to the case of the space $\overset{0}{W}_p^1$, how the question of finding the general form of a linear functional is reduced to theorems on solutions of integral identities.

Denote by $\mathbf{L}_p(\Omega)$ the space of vector-valued functions with n components, each of which belongs to the space $L_p(\Omega)$; we define the norm of a vector $A(x) \in \mathbf{L}_p(\Omega)$ by the formula

$$\|A\|_{L_p} = \left\{ \int_{\Omega} |A(x)|^p dx \right\}^{1/p}, \quad (11)$$

where $|A|$ is the length of the vector.

The correspondence $u \rightarrow \text{grad } u$ defines a linear isometric mapping of the space $\overset{0}{W}_p^1(\Omega)$ into $\mathbf{L}_p(\Omega)$; the image of $\overset{0}{W}_p^1(\Omega)$ under this mapping is a subspace of $\mathbf{L}_p(\Omega)$. If f is a linear functional on $\overset{0}{W}_p^1$, then, by the Hahn-Banach theorem, it extends to all of \mathbf{L}_p and, consequently, is determined by some vector-valued function $A(x) \in \mathbf{L}_q(\Omega)$ according to the formula

$$f(u) = \int_{\Omega} \overline{A(x)} \text{grad } u dx. \quad (12)$$

The vector $A(x)$ is not determined uniquely by the given functional. We shall show that among all vectors corresponding to the functional f , there is a unique one having the form $\text{grad } v$, where $v \in \overset{0}{W}_q^1(\Omega)$. Substituting in formula (12), instead of $A(x)$, the vector $\text{grad } v$, and comparing the results, we find that the function $v(x)$ must satisfy the integral identity

$$\int_{\Omega} \overline{\text{grad } v} \text{ grad } u \, dx = \int_{\Omega} \overline{A(x)} \text{ grad } u \, dx$$

for every $u(x) \in \overset{0}{W}_p^1(\Omega)$. From Theorem 3 follows the existence of a unique* function $v(x) \in \overset{0}{W}_q^1(\Omega)$ satisfying this identity. This completes the proof of Theorem 1 (for the case $l = 1$).

* It is easy to see that if two vectors $A_1(x)$ and $A_2(x)$ generate one and the same functional in $\overset{0}{W}_p^1$, then they correspond to one and the same function $v(x)$.

The fact that the norm in $[\overset{\circ}{W}_p^1]^*$ for $p \neq 2$ is only metrically equivalent, but does not coincide with the norm in $\overset{\circ}{W}_q^1$, is almost obvious. Indeed, let $v(x) \in \overset{\circ}{W}_q^1(\Omega)$. The unique (up to a numerical factor) element of the space $L_p(\Omega)$ for which

$$\int_{\Omega} A(x) \overline{\text{grad } v} \, dx = \|A(x)\|_{L_p} \|v(x)\|_{\overset{\circ}{W}_q^1},$$

is equal to $|\text{grad } v|^{q-2} \text{ grad } v$ and, generally speaking, is not the gradient of any function from $\overset{\circ}{W}_p^1(\Omega)$. But since the unit sphere in $\overset{\circ}{W}_p^1$ is weakly compact, there exists on it such an element $u(x)$ that

$$\int_{\Omega} \text{grad } u \overline{\text{grad } v} \, dx = \|v\|_{[\overset{\circ}{W}_p^1]^*}.$$

It follows from this that, as a rule,

$$\|v\|_{[\overset{\circ}{W}_p^1]^*} < \|v\|_{\overset{\circ}{W}_q^1}.$$

Leningrad State Pedagogical Institute
named after A. I. Herzen

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