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Abstract

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MATHEMATICS

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ON THE DIFFERENTIAL PROPERTIES OF THE SOLUTION OF THE DIRICHLET PROBLEM FOR DOMAINS WITH CORNERS

(Presented by Academician S. L. Sobolev on 24 II 1962)

In the present note the following question is solved: do there exist, or do there not exist, sufficiently smooth solutions, not written out explicitly, of the problems

$$\Delta u = 0 \quad \text{in } R, \tag{1}$$

$$u\big|_{AB} = P_{n_1}(s), \quad u\big|_{BC} = P_{n_2}(s), \quad u\big|_{CD} = P_{n_3}(s), \quad u\big|_{AD} = P_{n_4}(s);$$

$$\Delta v = P_m(x, y) \quad \text{in } R,$$

$$v\big|_{AB} = v\big|_{BC} = v\big|_{CD} = v\big|_{AD} = 0, \tag{2}$$

where AB, BC, CD, AD are the sides of the rectangle R; $P_m(x, y)$ is a polynomial of order m in x and y; $P_{n_i}(s)$ $(i = 1, 2, 3, 4)$ are polynomials of order n_i in the parameter s.

Let us note that if, in problems (1), (2), R is a domain with analytic boundary, then the solution of these problems has derivatives of every order (^{1}), although it is not always possible to write it out explicitly.

Let $n = \max_i n_i$, $1 \geqslant \alpha > 0$.

Theorem 1. If $u \in W_\infty^n H^\alpha(R)$ (^{2}), then u is a harmonic polynomial.

From Theorem 1 it follows directly that if $u \in W_\infty^n H^\alpha(R)$, then $n_1 = n_2 = n_3 = n_4$, and if $\min_i n_i = \bar{n} \neq n$, then $u \in W_\infty^r H^\alpha(R)$, where $r \leqslant \bar{n}$ for odd $\bar{n}$, and $r \leqslant \bar{n} + 1$ for even $\bar{n}$. Note also that for $u \in W_\infty^n H^\alpha(R)$, u will be a harmonic polynomial of order n for even n, while for odd n, u may be a harmonic polynomial of order $(n+1)$.

The proof of Theorem 1 is based on the results of S. M. Nikol' skii $(^3)$.

Theorem 2. If $v \in W_\infty^{m+2} H^\alpha(R)$, then v is a polynomial of order $(m+2)$.

Let the separate pieces of the piecewise smooth boundary Γ of an arbitrary domain Ω belong to the class $H_\infty^{r+2+1/p}$ $(^2)$ and be joined to one another at the points s_k by rectilinear ends at angles $\omega(s_k) = \pi/j_k$, where j_k are integers. Then from the results of V. V. Fufaev $(^4)$ the following theorem follows directly:

Theorem 3. In order that the solution of the problem

$$\Delta u = \varphi(x, y) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \Gamma$$

belong to the class $W_p^{(r)} H^\alpha(\Omega)$, it is necessary and sufficient that all particular solutions $u_\xi \in W_p^{(r_1)} H^{\alpha_1}(\Omega)$, where $r + \alpha \leqslant r_1 + \alpha_1$, of the equation $\Delta u_\xi = \varphi(x, y)$.

where $\bar{\varphi}(x, y)$ in Ω coincides with $\varphi(x, y)$, satisfying, for all $\gamma = 0, 1, \ldots$ for which $\gamma j_k < r + \alpha - 1/p$, the following relations on the boundary Γ:

$$u_\xi^{(\gamma j_k)}[x_q(s_k+0),\, y_q(s_k+0)] = (-1)^\gamma u_\xi^{(\gamma j_k)}[x_v(s_k-0),\, y_v(s_k-0)],$$

where $x_q(s), y_q(s)$ and $x_v(s), y_v(s)$ are the equations of the separate pieces of the boundary, and the derivatives $u_\xi^{(\gamma j_k)}$ with respect to the parameter s are taken along the rectilinear segments forming the angle ω/j_k.

In addition to Theorem 3, in the proof of Theorem 2 the following easily verified propositions are used:

1. A particular solution of the equation $\Delta u = a x^m y^n$, where m and n are positive integers and a is a constant, will be

$$u = \frac{a(-1)^{(n-1)/2}}{(m+2)(m+1)C_{m+n+2}^n} P_{m+n+2,s}, \qquad n \text{ odd};$$

$$u = \frac{a(-1)^{n/2}}{(m+2)(m+1)C_{m+n+2}^n} P_{m+n+2,s}, \qquad n \text{ even},$$

where

$$P_{i,j} = \sum_{r=0}^{j/2} (-1)^r C_i^{2r} x^{i-2r} y^{2r}, \qquad j \text{ even};$$

$$P_{i,j} = \sum_{r=0}^{(j-1)/2} (-1)^r C_i^{2r+1} x^{i-2r-1} y^{2r+1}, \qquad j \text{ odd}.$$

2. Let $\{P_1(x,y), P_2(x,y), \dots, P_s(x,y)\}$ be a linearly independent system of polynomials. In order that the system $\{\Delta P_1, \Delta P_2, \dots, \Delta P_s\}$ be linearly independent, it is necessary and sufficient that the system $\{P_1, P_2, \dots, P_s, P_r\}$, where P_r is a harmonic polynomial, also be linearly independent.

If in (2) $m = 0$ or $m = 1$, then $v \in W_\infty^1 H^\alpha(R)$, where α is any number less than one. For $m = 2$, in order that $v \in W_\infty^1 H^\alpha(R)$, where $\alpha > 0$, it is necessary, as follows from Theorem 3, that $P_m(x,y) \equiv c[x^2 - ax + y^2 - by]$, where a and b are the sides of the rectangle R, and c is an arbitrary constant; but then, as follows from Theorem 2,

$$v = \frac{c}{2}[x, y(x-a)(y-b)].$$

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CITED LITERATURE

1. N. M. Günter, *Potential Theory and Its Applications to Basic Problems of Mathematical Physics*, Moscow, 1953.
2. S. M. Nikol' skii, *Matem. sbornik*, 33 (75), No. 2, 261 (1953).
3. S. M. Nikol' skii, *Matem. sbornik*, 43 (85), No. 1 (1957).
4. V. V. Fufaev, DAN, 131, No. 1, 37 (1960).

Note: Figure translations are in progress. See original paper for figures.

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