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Abstract

Full Text

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STRICTLY RELATED MODELS AND RECURSIVELY PERFECT ALGEBRAS

The present note arose in connection with the following question of A. Mostowski, formulated in the book ⁽¹⁾ on p. 84. Consider the arithmetic

$$S = \langle \{0, 1, 2, \dots\}, +, \times \rangle$$

and ask whether it is possible to define on the natural number series a binary operation $*$ and to fix natural numbers a_1, \dots, a_p in such a way that the following conditions are simultaneously satisfied: 1) the natural number series must be a group with respect to the operation $*$; 2) the relation $x * y = z$ must be representable by a formula of the restricted predicate calculus on S ; 3) the relations $x + y = z$ and $x \times y = z$ must be representable by formulas of the restricted predicate calculus on the group

$$\langle \{0, 1, 2, \dots\}, *, a_1, \dots, a_p \rangle$$

with distinguished elements a_1, \dots, a_p . Below a solution is given of a general problem related to the problem of A. Mostowski just mentioned. From this solution, as an accompanying result, there also follows a positive answer to A. Mostowski's question.

1. Strict relatedness of models. Let $\mathbf{K}_1, \mathbf{K}_2$ be arbitrary classes of models of signatures, respectively,

$$\sigma_1 = \{P_1, \dots, P_s, a_1, \dots, a_p\}$$

and

$$\sigma_2 = \{Q_1, \dots, Q_t, b_1, \dots, b_q\}.$$

By \mathfrak{F}_i ($i = 1, 2$) denote the totality of all formulas of the restricted predicate calculus having signature σ_i . By $\Sigma(\mathbf{K}_i)$ denote the subset of those closed formulas from \mathfrak{F}_i which are true on every model of the class \mathbf{K}_i . Let $\mathfrak{A}(x)$ be some formula from \mathfrak{F}_2 , containing only one free object variable x . To assign a homomorphism φ of the set \mathfrak{F}_1 into \mathfrak{F}_2 means to associate with each predicate symbol $P_i(x_1, \dots, x_{n_i})$ a definite formula

$$\mathfrak{B}_i(x_1, \dots, x_{n_i})$$

from \mathfrak{F}_2 , and with each individual object symbol a_j to associate some formula $\mathfrak{A}_j(x)$ from \mathfrak{F}_2 with one free object variable x . If \mathfrak{H} is a formula from \mathfrak{F}_1 , then

by \mathfrak{H}^φ we shall denote the formula from \mathfrak{F}_2 obtained from \mathfrak{H} by the following transformations: 1) instead of $P_i(\mathbf{a}_1, \dots, \mathbf{a}_{n_i})$ in \mathfrak{H} , everywhere write

$$\mathfrak{P}_i(\mathbf{a}_1, \dots, \mathbf{a}_{n_i});$$

2) the quantifiers in \mathfrak{H} are restricted to the set \mathfrak{R} of those elements x for which $\mathfrak{R}(x)$ is true; 3) if, in the formula \mathfrak{H}_1 obtained after transformations 1), 2), there are individual signs a_1, \dots, a_p , then in place of \mathfrak{H}_1 write the formula

$$(\exists a_1 \dots a_p)(\mathfrak{H}_1 \ \& \ \mathfrak{A}_1(a_1) \ \& \ \dots \ \& \ \mathfrak{A}_p(a_p)),$$

which will be the required formula \mathfrak{H}^φ .

The homomorphism φ of \mathfrak{F}_1 into \mathfrak{F}_2 is called a relative \mathfrak{R} -interpretation of \mathbf{K}_1 in \mathbf{K}_2 (see (1)), if

$$\Sigma(\mathbf{K}_1)^\varphi \subseteq \Sigma(\mathbf{K}_2).$$

A relative \mathfrak{R} -interpretation is called simply an interpretation if $\mathfrak{R}(x)$ is identically true on \mathbf{K}_2 . The classes of models $\mathbf{K}_1, \mathbf{K}_2$ are called related if there exist an interpretation φ of \mathbf{K}_1 in \mathbf{K}_2 and an interpretation ψ of \mathbf{K}_2 in \mathbf{K}_1 such that

$$(\mathfrak{H}^{\varphi\psi} \leftrightarrow \mathfrak{H}) \in \Sigma(\mathbf{K}_1), \quad (\mathfrak{G}^{\psi\varphi} \leftrightarrow \mathfrak{G}) \in \Sigma(\mathbf{K}_2) \quad (\mathfrak{H} \in \mathfrak{F}_1, \ \mathfrak{G} \in \mathfrak{F}_2); \quad (1)$$

$$(x = a_j \leftrightarrow \mathfrak{A}_j(x)^\psi) \in \Sigma(\mathbf{K}_1) \quad (x = b_j \leftrightarrow \mathfrak{B}_j(x)^\varphi) \in \Sigma(\mathbf{K}_2). \quad (2)$$

Let φ be an \mathfrak{R} -interpretation of \mathbf{K}_1 in \mathbf{K}_2 ,

$$\mathfrak{N} = \langle N; Q_1, \dots, Q_t, b_1, \dots, b_q \rangle$$

be some \mathbf{K}_2 -model. The model

$$\mathfrak{N}^\varphi = \langle \mathfrak{R}; \mathfrak{P}_1, \dots, \mathfrak{P}_s, \mathfrak{A}_1, \dots, \mathfrak{A}_p \rangle,$$

where \mathfrak{R} is the totality of those elements of N for which $\mathfrak{R}(x)$ is true, we agree to denote by \mathfrak{N}^φ . The \mathfrak{R} -interpretation φ we agree to call

to be called **isomorphic** if for every model \mathfrak{M} from K_1 in K_2 there exists a model \mathfrak{N} such that \mathfrak{N}^φ is isomorphic to \mathfrak{M} . The classes K_1 and K_2 will be called **strictly related** if there are isomorphic interpretations φ of K_1 in K_2 and interpretations ψ of K_2 in K_1 satisfying conditions (1), (2).

The notions set forth will be applied below to the case in which K_1 and K_2 contain only one model each.

Sec. 2. Recursively perfect algebras. A one-to-one mapping α of a certain set of natural numbers D_α onto the universe M of the model

$$\mathfrak{M} = \langle M; P_1, \dots, P_s, a_1, \dots, a_p \rangle$$

is called a numbering of \mathfrak{M} . The numbering α is called recursive (2) if D_α is recursive and all predicates P_1, \dots, P_s under α are transformed into recursive

relations on D_α . We shall call a model \mathfrak{M} **recursively stable** if all its recursive numberings are recursively equivalent to the numbering α . In ⁽²⁾ it is shown that every recursively numbered finitely generated algebra is recursively stable. However, an algebra not having a finite number of generators may also possess recursive stability.

Theorem 1. *Every field K that is a finite extension of the field of rational numbers, every group $SL(n, K)$ over such a field K for $n \geq 2$, and also the group $RSL(n, K)$ of all triangular matrices from $SL(n, K)$, and every complete nilpotent torsion-free group of finite rank, are recursively numberable recursively stable algebras.*

The proof is easily carried out directly for fields K and nilpotent groups. The main stages of the proof for the groups $SL(n, K)$ and $RSL(n, K)$ are indicated below in Sec. 3.

A recursively numbered model or algebra is called **recursively perfect** if it is infinite and every recursive predicate defined on this model is representable by a formula of the restricted predicate calculus. Gödel's theorem ⁽¹⁾ shows that arithmetic S is a recursively perfect algebra. The definition also immediately implies

Theorem 2. *Any two recursively perfect models are strictly related to each other.*

A relative \mathfrak{R} -interpretation φ of the model

$$\mathfrak{M} = \langle M; P_1, \dots, P_s, a_1, \dots, a_p \rangle$$

in the model

$$\mathfrak{N} = \langle N; Q_1, \dots, Q_t, b_1, \dots, b_q \rangle,$$

which has a recursive numbering β , will be called **recursive** if the relations

$$P_i^\varphi = \mathfrak{P}_i(x_1, \dots, x_{n_i}) \quad (i = 1, \dots, s)$$

and $\mathfrak{R}(x)$ are recursive in the numbering β .

Theorem 3. *Suppose there exists a relative isomorphic and recursive interpretation φ of a recursively perfect model*

$$\mathfrak{M} = \langle M; P_1, \dots, P_s \rangle$$

in a recursively numbered model

$$\mathfrak{N} = \langle N; Q_1, \dots, Q_t, b_1, \dots, b_q \rangle,$$

and suppose there exists a formula of first degree

$$\mathfrak{U}(x; x_1, \dots, x_r),$$

representing a one-to-one recursive mapping of a certain set \mathfrak{S} of sequences $\langle x_1, \dots, x_r \rangle$ of elements of \mathfrak{R} onto the whole model \mathfrak{N} . Then the model \mathfrak{N} is strictly related to \mathfrak{M} . If, moreover, the model \mathfrak{N} is recursively stable, then it is also perfect.

Proof. Let

$$P_i^\varphi = \mathfrak{P}_i \quad (i = 1, \dots, s).$$

The model

$$\mathfrak{M}_0 = \langle \mathfrak{R}; \mathfrak{P}_1, \dots, \mathfrak{P}_s \rangle$$

is recursive and abstractly isomorphic to the perfect model \mathfrak{M} . Therefore the model \mathfrak{M}_0 is also perfect. The set \mathfrak{S} is recursively enumerable. Therefore there exists on \mathfrak{R} a recursive relation

$$\mathfrak{B}(x; x_1, \dots, x_r),$$

which maps \mathfrak{R} one-to-one onto \mathfrak{S} . Since the model \mathfrak{M}_0 is perfect, the relation \mathfrak{B} is representable by a formula of the restricted predicate calculus. But then the formula

$$\mathfrak{W}(x, y) \stackrel{df}{=} (\exists x_1, \dots, x_r) \left(\bigwedge_{\lambda} \mathfrak{R}(x_\lambda) \ \& \ \mathfrak{B}(x; x_1, \dots, x_r) \ \& \ \mathfrak{U}(y; x_1, \dots, x_r) \right)$$

represents a one-to-one recursive mapping of \mathfrak{R} onto N . Con-
assuming, by definition,

$$P_i^\psi \stackrel{df}{=} P'_i(x_1, \dots, x_{n_i}) \stackrel{df}{=} (\exists y_1 \dots y_{n_i}) (\mathfrak{B}(y_1, x_1) \ \& \ \dots \\ \dots \ \& \ \mathfrak{B}(y_{n_i}, x_{n_i}) \ \& \ \mathfrak{P}_i(y_1, \dots, y_{n_i})),$$

we see that ψ is an isomorphic and recursive interpretation of \mathfrak{M} on the model \mathfrak{N} . Therefore the model $\mathfrak{M}_1 = \langle N; P'_1, \dots, P'_s \rangle$ is perfect. Recursive predicates Q_i are defined on the model \mathfrak{M}_1 . By the perfection of \mathfrak{M}_1 , the predicates Q_i are formally expressible through P'_i .

Thus, the models \mathfrak{M} and \mathfrak{N} are strictly related and, moreover, on \mathfrak{N} every recursive relation is formally representable. Therefore, if it is known that \mathfrak{N} is recursively stable, then it will also be recursively perfect.

Corollary. *The ring of rational integers, as well as any finite extension of the field of rational numbers having no nontrivial automorphisms, are recursively perfect algebras.*

Indeed, if K is a finite extension of the field of rational numbers and θ is a primitive element of K , then, according to J. Robinson ⁽³⁾, there exists a formula $\mathfrak{R}(x)$ selecting in K the rational numbers, while the formula

$$\mathfrak{U}(x; x_1, \dots, x_n) \stackrel{df}{=} (\exists \theta)(x = x_1 + x_2\theta + \dots + x_n\theta^{n-1} \ \& \ f(\theta) = 0),$$

gives a one-to-one mapping of \mathfrak{R} onto the set of systems $\langle x_1, \dots, x_n \rangle$ of rational numbers (f is an irreducible polynomial whose root is θ , and n is the degree of f).

§ 3. Linear groups.

According to a remark of A. Mostowski ⁽¹⁾, the automorphism groups of strictly related models are isomorphic. Recursively perfect algebras have no identical automorphisms, whereas all infinite groups have them. Therefore only groups with at least two distinguished elements can be recursively perfect; and such groups, as the following theorem shows, do indeed exist.

Theorem 4. *The groups $SL(n, K)$ and $RSL(n, K)$ over a recursively perfect field K , with distinguished matrices A, B , are recursively perfect for $n \geq 3$.*

Here $SL(n, K)$ is the group of matrices of order n with determinant one and entries from K ; $RSL(n, K)$ is the group of all triangular matrices from $SL(n, K)$; A is the Jordan cell with ones on the diagonal; B is the transposed cell for A in the case of the groups $SL(n, K)$, and B is an arbitrary diagonal matrix of “general” form from $SL(n, K)$ in the case when the groups $RSL(n, K)$ are considered.

We shall indicate the general course of the proof for the groups $RSL(3, K)$ and $SL(3, K)$. Similar arguments also apply to matrix groups of higher order. The recursive numbering of the field K naturally induces a recursive numbering of the group $SL(3, K)$, which we shall have in mind below.

First consider the group $RSL(3, K)$ with fixed matrices

$$A = e_{11} + e_{22} + e_{33} + e_{12} + e_{23}$$

and

$$B = b_1 e_{11} + b_2 e_{22} + b_3 e_{33}, \quad \text{where } b_1 b_2^{-1} \neq b_2 b_3^{-1}.$$

Putting

$$\mathfrak{R}(X) \stackrel{df}{=} (\exists Y)(AY = YA \ \& \ YBAB^{-1} = BAB^{-1}Y \ \& \ Y^3 = X),$$

$$\mathfrak{R}_1(X) \stackrel{df}{=} X^2 = E \ \& \ X \neq E \ \& \ XA = AX,$$

$$\mathfrak{R}_2(X) \stackrel{df}{=} (\exists Y)(\mathfrak{R}_1(Y) \ \& \ X = A^{-1}YA^{-1}Y),$$

we see that \mathfrak{R} consists of matrices of the form $E + ae_{13}$, while \mathfrak{R}_2 contains only the matrix $E + e_{13}$, where $E = e_{11} + e_{22} + e_{33}$. Now putting (see (4))

$$Z = X \oplus Y \stackrel{df}{=} Z = XY,$$

$$\begin{aligned} Z = X \circ Y &\stackrel{df}{=} (\exists UVW)(UB = BU \ \& \ VB = BV \ \& \ \mathfrak{R}_2(W) \ \& \ UW = \\ &= XU \ \& \ VW = YV \ \& \ UVW = ZUV), \end{aligned}$$

we obtain a recursive \mathfrak{R} -interpretation of \mathbf{K} in $RSL(3, \mathbf{K})$ with the isomorphism $a \rightarrow E + ae_{13}$ of the field \mathbf{K} onto the field $\langle \mathfrak{R}; \oplus, \circ \rangle$.

Using certain particular properties of triangular matrices, it is not hard to construct a formula $\mathfrak{U}(X; X_{11}, X_{22}, X_{33}, X_{12}, X_{13}, X_{23})$, true only for matrices of the form $X_{ij} = E + a_{ij}e_{ij}$, $X = \sum a_{ij}e_{ij}$ ($i \leq j$, $\prod a_{ii} = 1$), and, consequently, mapping the group $RSL(3, \mathbf{K})$ one-to-one onto the collection of sequences $\langle X_{11}, X_{22}, X_{33}, X_{13}, X_{12}, X_{23} \rangle$ of elements of the set \mathfrak{R} .

In order to satisfy the conditions of Theorem 3 and thereby prove the perfection of the group $RSL(3, \mathbf{K})$, it remains only to prove the recursive stability of this group. The recursive numeration of the field \mathbf{K} induces a natural recursive numeration α of the group $RSL(3, \mathbf{K})$. Let β be any other recursive numeration of this group. From the form of the formula $\mathfrak{R}(x)$ and of the formulas for \oplus and \circ , we easily conclude that the set \mathfrak{R} and the operations \oplus, \circ are recursive in any recursive numeration of $RSL(3, \mathbf{K})$. Since the model $\langle \mathfrak{R}; \oplus, \circ \rangle$ is recursively perfect, the numerations α and β on it are recursively equivalent, i.e. there exists an algorithm \mathcal{A} which, from the number of any element of \mathfrak{R} in one of the numerations α, β , finds its number in the other numeration. From the construction of the above-mentioned formula \mathfrak{U} one can extract an algorithm \mathcal{B} for finding the number of a matrix X from the numbers of its "coordinate" matrices X_{ij} ($i \leq j$) in any recursive numeration of the group under consideration. Now, knowing the α -number of some matrix X of this group, we find the α -numbers of the matrices X_{ij} by means of the algorithm \mathcal{B} . Using the algorithm \mathcal{A} , we find the β -numbers of the matrices X_{ij} , and by means of the algorithm \mathcal{B} we find the β -number of X . Thus the recursive perfection of the group $RSL(3, \mathbf{K})$ with distinguished elements A, B is proved.

Let us now consider the group $SL(3, \mathbf{K})$ with distinguished matrices $A = E + e_{12} + e_{23}$ and $A' = E + e_{21} + e_{32}$. The formulas

$$\mathfrak{P}(X) \stackrel{df}{=} XAX^{-1}A = AXAX^{-1}, \quad \mathfrak{P}'(X) \stackrel{df}{=} XA'X^{-1}A' = A'XA'X^{-1}$$

single out in $SL(3, \mathbf{K})$, respectively, the subgroup G_1 of upper triangular matrices and the subgroup G_2 of lower triangular matrices whose diagonal satisfies a certain condition. The formula $\mathfrak{P}(X)\mathfrak{P}'(X)X^6A \neq AX^6$ singles out in $SL(3, \mathbf{K})$ the collection of diagonal matrices of the form $xe_{11} + axe_{22} + a^2xe_{33}$, where $a^6 \neq 1$, $x^3a^3 = 1$. Any matrix permutable with a matrix of the indicated form, but not itself having this form and different from E , can be taken as the matrix B for the construction in the groups G_1 and G_2 , with the aid of the matrices A, A' , of "coordinatizing" formulas $\mathfrak{U}_1(X; X_{11}, X_{22}, X_{33}, X_{12}, X_{13}, X_{23})$ and $\mathfrak{U}_2(X; X_{11}, X_{22}, X_{33}, X_{21}, X_{31}, X_{32})$. Finally, using the fact that every matrix of $SL(3, \mathbf{K})$ is representable in the form $Y_1Y_2Y_3Y_4Y_5$ ($Y_1, Y_3, Y_5 \in G_1$; $Y_2, Y_4 \in G_2$) and knowing the formulas \mathfrak{U}_1 and \mathfrak{U}_2 , it is easy to construct a coordinatizing formula for the group $SL(3, \mathbf{K})$ itself. The recursive stability of $SL(3, \mathbf{K})$ is easily obtained from the recursive stability of the subgroups G_1 and G_2 .

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