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`![Fig. 1](figure)`

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MECHANICS

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LINEARLY ENVELOPING CYCLOIDAL CURVES

We shall call *linearly enveloping cycloidal curves* those curves obtained as the envelopes of the positions of a straight line belonging to a circle rolling without slipping along a fixed circle or a straight line. This class of curves can be reproduced by three-membered satellite mechanisms. Consider a mechanism (Fig. 1) consisting of a fixed circular wheel 1, a moving circular wheel 2 (satellite), and a link 3, entering into pairs of the fifth class with wheels 1 and 2. Let an arbitrary straight line u—u be chosen in the plane of wheel 2, the shortest distance BN of which from point B is equal to $BN = a$. We shall find the equations of the curve linearly enveloping the positions of the straight line u—u when wheel 2 rolls along wheel 1. Rotate link 3 through the angle φ into position $3'$. Then wheel 2 will rotate through the angle ψ and pass into position $2'$, the straight line u—u into position u'—u', and the points N and P into positions N' and P'. Drop from point A the perpendicular AD to the straight line u'—u'. The equation of the family of straight lines u—u will be

Fig. 1

$$x \cos\theta + y \sin\theta = p, \tag{1}$$

where $p = AD$ and θ is the angle formed by the straight line AD with the axis Ax.

The segment p is equal to

$$p = -(R + r)\cos\psi + a, \tag{2}$$

where R is the radius of wheel 1, and r is the radius of wheel 2.

The angle Ψ of rotation of the straight line u—u is equal to

$$\Psi = \psi + \varphi = (1 + \lambda)\psi, \tag{3}$$

since $\varphi = \frac{r}{R}\psi = \lambda\psi$, where $\lambda = r/R$.

From Fig. 1 it follows that

$$\theta + \Psi = 180^\circ + \alpha = \beta. \tag{4}$$

From equations (4), taking condition (3) into account, we obtain

$$\psi = \frac{\beta - \theta}{1 + \lambda}. \tag{5}$$

Then equation (2) takes the form

$$p = -(R + r)\cos\frac{1}{1 + \lambda}(\beta - \theta) + a. \tag{6}$$

Substituting the obtained expressions for p into equation (1), we obtain for the family of straight lines u—u the equation

$$x\cos\theta + y\sin\theta = -(R + r)\cos\frac{1}{1 + \lambda}(\beta - \theta) + a. \tag{7}$$

Let us find the partial derivative with respect to the parameter θ of expression (1). We have

$$y\cos\theta - x\sin\theta = \frac{\partial p}{\partial\theta}, \tag{8}$$

or, taking equation (6) into account,

$$y\cos\theta - x\sin\theta = -(R + r)\frac{1}{1 + \lambda}\sin\frac{1}{1 + \lambda}(\beta - \theta). \tag{9}$$

From equations (7) and (9) we obtain the parametric equations for the sought linearly enveloping curve in the form $x = x(\theta)$ and $y = y(\theta)$. We have

$$x = -(R + r)\left[\cos\frac{1}{1 + \lambda}(\beta - \theta)\cos\theta - \frac{1}{1 + \lambda}\sin\frac{1}{1 + \lambda}(\beta - \theta)\sin\theta\right] + a\cos\theta; \tag{10}$$

$$y = -(R + r)\left[\cos\frac{1}{1 + \lambda}(\beta - \theta)\sin\theta + \frac{1}{1 + \lambda}\sin\frac{1}{1 + \lambda}(\beta - \theta)\cos\theta\right] + a\sin\theta. \tag{11}$$

Fig. 2

Figure 2: Fig. 2

It follows from equations (10) and (11) that the linearly enveloping curve $x = x(\theta)$ and $y = y(\theta)$ is a transcendent curve of the cycloidal-curve type. It is also not difficult to see that any straight line belonging to the plane of wheel 2 and parallel to the straight line u—u will have as its linearly enveloped curve the curve determined by equations (10) and (11) with the corresponding value of the parameter a. Therefore, for the study of these curves it is sufficient to consider the case when the straight line u—u coincides with the diameter of wheel 2, i.e., when the parameter $a = 0$. The construction of the linearly enveloping curve can be carried out point by point. For this purpose, from point P_{21} (Fig. 2), which is the instantaneous center in the motion of wheel 2 relative to wheel 1, we drop the perpendicular $P_{21}M$ to the straight line u—u. Point M will belong to the linearly enveloping curve in the position under consideration of wheel 2 and of the straight line u—u. The functions $p = p(\theta)$ and $dp/d\theta = f(\theta)$, determined by equations (6) and (8), can be represented geometrically. For this purpose, let us attach to the mechanism (see Fig. 2) a two-slider group consisting of link 4, rotating about the fixed axis A, and a cross-shaped slider 5 with mutually perpendicular axes of the guides, sliding along the directions AT and u—u. From point $B(b)$ we draw the straight line BD, connecting point B with point D of intersection of the directions AT and u—u. We extend the straight line BD until it intersects at point d with the direction Ad, parallel to the direction $P_{21}D$. The segment (pd) will represent the velocity vector $\vec{v}_D$ of point D, belonging to the straight line u—u, rotated through an angle of 90°. Through point d we draw a parallel to the direction AT, and through point A a parallel to the straight line u—u. The segment (pt), rotated through an angle of 90°, will be proportional to the rate of change of the vector $|AD| = p$, i.e., will be proportional to $\partial p/\partial\theta$. It is now not difficult to see

Fig. 2

that equation (6) will be the polar equation of the podera of the linearly enveloping curve $x = x(\theta)$ and $y = y(\theta)$, if point A is chosen as the pole of the podera. This podera will be the locus of the points D (Fig. 2) of slider 5.

If the moving wheel 2 (Fig. 3), with the straight line $u - u$ belonging to it, has internal contact with the fixed wheel 1, then the parametric equations of the linearly enveloping curve $x = x(\theta)$ and $y = y(\theta)$ will be

$$x = (R - r)\left[\cos\frac{1}{1-\lambda}(\beta-\theta)\cos\theta - \lambda\frac{1}{1-\lambda}\sin\frac{1}{1-\lambda}(\beta-\theta)\sin\theta\right] + a\cos\theta; \quad (12)$$

Fig. 3

Figure 3: Fig. 3

Fig. 4

Figure 4: Fig. 4

$$y = (R - r)\left[\cos\frac{1}{1-\lambda}(\beta-\theta)\sin\theta + \frac{1}{1-\lambda}\sin\frac{1}{1-\lambda}(\beta-\theta)\cos\theta\right] + a\sin\theta, \quad (13)$$

where $\lambda = r/R$. In Fig. 3 the angles $\varphi, \psi, \Psi, \theta$ and β are shown, as well as the position of the point M belonging to the linearly enveloping curve. The derivation of equations (12) and (13) is analogous to the derivation of equations (10) and (11).

If in equations (12) and (13) we put $\beta = 135^\circ$ and $\lambda = \frac{1}{2}$, then these equations reduce to the equation

$$x^{2/3} + y^{2/3} = a^{2/3},$$

where $a = 2(R - r)$, i.e., in this case the linearly enveloping curve will be an astroid.

Fig. 3

Let us consider the case when the moving wheel 2 (Fig. 4) is rolled without slipping along the fixed link 1, which is a straight line $x - x$. The connecting rod 3 then slides in fixed guides A, the axis of which is parallel to the axis $x - x$. Let an arbitrary straight line $u - u$ be chosen in the plane of wheel 2, the shortest distance BN of which from point B is equal to $BN = a$. To compose the equations of the curve linearly enveloping the straight line $u - u$, move the point B of connecting rod 3 and of wheel 2 into the position B'.

Fig. 4

Then wheel 2 will move into position $2'$, and the straight line $u - u$ will occupy the position $u' - u'$. The angle of rotation ψ of wheel 2 will be related to the displacement $s = BB'$ of the center B of wheel 2 by the condition $\psi = s/r$, where r is the radius of wheel 2.

Drop from the point P the perpendicular $PD = p$ to the line $u' - u'$. Then the equation of the family of lines $u - u$ will have the form

$$x\cos\theta + y\sin\theta = r\left[\sin\theta + \left(\frac{4\pi}{3} - \theta\right)\cos\theta\right] + a, \quad (14)$$

since

$$p = PD = DC - PC = (a - r\cos\psi) + \psi r\cos\theta, \qquad \psi = 4\pi/3 - \theta.$$

Next we find the partial derivative with respect to the parameter θ of expression (14). We obtain

$$y\cos\theta - x\sin\theta = -r\left(\frac{4\pi}{3} - \theta\right)\sin\theta. \tag{15}$$

From equations (14) and (15) we obtain parametric equations for the desired linearly enveloping curve in the form $x = x(\theta)$ and $y = y(\theta)$. We have

$$x = r\left[\sin\theta\cos\theta + \left(\frac{4\pi}{3} - \theta\right)\right] + a\cos\theta; \tag{16}$$

$$y = r\sin^2\theta + a\sin\theta. \tag{17}$$

From equations (16) and (17) it follows that this linearly enveloping curve is also a transcendental curve of the type of cycloidal curves. The construction of this linearly enveloping curve can be carried out point by point. In Fig. 4 the point M of this curve is shown in the position when the curve $u - u$ occupies the position $u' - u'$. To find the point M, it is necessary to drop a perpendicular from the instantaneous center of rotation P_1 of wheel 2 to the line $u' - u'$.

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Note: Figure translations are in progress. See original paper for figures.