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Abstract

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SOME ESTIMATES NEAR THE BOUNDARY OF A DOMAIN FOR A POLYHARMONIC FUNCTION AND ITS DERIVATIVES DEFINED IN AN N-DIMENSIONAL DOMAIN

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In the present paper, estimates will be obtained for a polyharmonic function and its partial derivatives of arbitrary order in the L_2 metric in an arbitrary N-dimensional domain G with boundary Γ of class $C^{(2)}$. The investigations are based on the estimates obtained in the papers ([1,2]) and are carried out essentially in the three-dimensional case. They are transferred to the N-dimensional case by analogy.

§ 1. Let $N = 3$ and

$$\Delta^l u = 0, \tag{1,1}$$

$l > 0$ an integer, and let

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} u^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi < +\infty; \tag{1,2}$$

$$I_r^{(q)} = \int_0^{2\pi} \int_0^{\pi} u^{(q)2} r^2 \sin\theta \, d\theta \, d\varphi, \tag{1,3}$$

where $u^{(q)}$ is any mixed derivative taken of the function u with respect to the variables r, θ, φ; $q > 0$ is an integer.

In the proposed paper, first, the inequality will be proved

$$I_r^{(q)} \leqslant \frac{C_{l,q}}{(R-r)^{2q+1}} \, I. \tag{1,4}$$

Next it is proved that if u is an l-harmonic function defined in a three-dimensional domain G with boundary $\Gamma \in C^{(2)}$, then there exists a piecewise smooth (consisting of pieces of spheres) closed (geometric) surface $\gamma_\delta \subset G$, possessing the following properties:

1) The distance $\rho(\bar{x}, \Gamma)$ from each point $\bar{x} \in \gamma_\delta$ to Γ satisfies the inequalities

$$ {}^1/_3\delta \leqslant \rho(\bar{x}, \Gamma) \leqslant {}^1/_2\delta \quad (0 < \delta < \delta_0). \tag{1,5} $$

2) For each point $P \in \Gamma$ there exists a unique point $Q \in \gamma_\delta$ lying on the inner normal to Γ drawn from P, and γ_δ contains no other points except

$$ Q = \varphi(P). \tag{1,6} $$

3) The inequality holds

$$ \mu_1 \, d\Gamma \leqslant d\gamma_\delta \leqslant \mu_2 \, d\Gamma \tag{1,7} $$

(μ_1, μ_2 are positive constants), where $d\gamma_\delta$ is the differential near the point Q of any smooth piece of the surface γ_δ; $d\Gamma$ is the differential of the surface near the point P.

surface near the point P, corresponding to Q (see formula (1.6)). In this case the inequality

$$ I_{\gamma_\delta}^{(q)} \leqslant \frac{C_{l,q}^*}{\delta^{2q+1}} \, I_0, \tag{1.8} $$

holds, where

$$ I_{\gamma_\delta}^{(q)} = \iint_{\gamma_\delta} u^{(q)2} \, d\gamma_\delta; \tag{1.9} $$

$$ I_0 = \iiint_G u^2 \, dG. \tag{1.10} $$

§ 2. The proof of estimates of the form (1.4) in the case when the q-th derivatives are computed with respect to the variables r and φ reduces directly to inequalities proved in paper (1). What remains to be considered is the proof concerning derivatives with respect to the variable θ.

Lemma. Let $f(r)$ be a positive polynomial of degree $4(l-1)$. Then for all natural $n \geqslant 0$ the inequality

$$ \int_0^1 r^{2n+2} f(r) \, dr \geqslant \varkappa_l \int_0^1 r^{2n+1} f(r) \, dr, \tag{2.1} $$

holds, where $\varkappa_l$ is a positive constant depending only on l.

Let a harmonic function be given in the unit ball ($N = 3,\ 0 \leqslant r < 1$):

$$v = \sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} A_{nk} \cos k\varphi \cdot P_n^{(k)}(\cos\theta).$$

By virtue of the work of S. M. Nikol' skii ([2]), we shall have

$$\int_0^{2\pi}\int_0^{\pi}\left\{\sum_{n=0}^{\infty} r^n \sum_{k=0}^{n} A_{nk}\cos k\varphi \left[P_n^{(k)}(\cos\theta)\right]_\theta^{(q)}\right\}^2 r\sin\theta\, d\theta\, d\varphi \leqslant$$

$$\leqslant \frac{C_q}{(1-r)^{2q+1}} \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{A_{nk}^2 (n+k)!}{(n+1)^2 (n-k)!}. \tag{2.2}$$

Let us note that inequality (2.2) is valid for arbitrary numbers A_{nk} independent of φ and θ, and for any r ($0 \leqslant r < 1$).

Put

$$A_{nk}^2 = F_{nk} = \left[a_{1n}^{(k)} + a_{2n}^{(k)} r^2 + \dots + a_{ln}^{(k)} r^{2(l-1)}\right]^2, \tag{2.3}$$

where $a_{in}^{(k)}$ ($i = 1, 2, \dots, l$) are linear combinations of the Fourier coefficients of the boundary values of the l-harmonic function u on the unit ball.

Next consider

$$\Phi_{nk} = \int_0^1 r^{2n+1}\left[a_{1n}^{(k)} + a_{2n}^{(k)} r^2 + \dots + a_{ln}^{(k)} r^{2(l-1)}\right]^2 dr, \tag{2.4}$$

$$\Phi_{nk}^* = \int_0^1 r^{2n+2}\left[a_{1n}^{(k)} + a_{2n}^{(k)} r^2 + \dots + a_{ln}^{(k)} r^{2(l-1)}\right]^2 dr. \tag{2.5}$$

By virtue of the lemma we have

$$\Phi_{nk} < \varkappa_l \Phi_{nk}^*. \tag{2.6}$$

In paper ([1]) the relation

$$|F_{nk}| \leq C_l \Phi_{nk}(n+1), \tag{2,7}$$

valid for all n and k ($C_l > 0$ is a constant), was proved.
By virtue of (2,2), (2,3), (2,6), (2,7), and the known representation of a harmonic function in a three-dimensional ball through $a_{ln}^{(k)}$, we shall have

$$I_r^{(q)} = \int_0^{2\pi}\int_0^{\pi}\left\{\sum_{n=0}^{\infty} r^n \sum_{k=0}^{n}\sqrt{|F_{nk}|}\cos k\varphi\left[P_n^{(k)}(\cos\theta)\right]_\theta^{(q)}\right\}^2 r\sin\theta\, d\theta\, d\varphi \leq$$
$$\leq \frac{C_q}{(1-r)^{2q+1}}\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{F_{nk}(n+k)!}{(n+1)^2(n-k)!} \leq \frac{C_l C_q}{(1-r)^{2q+1}}\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\Phi_{nk}(n+1)!}{(n+1)(n-k)!} <$$
$$< \frac{C_{l,q}}{(1-r)^{2q+1}}\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\Phi^*_{nk}(n+k)!}{(n+1)(n-k)!} = \frac{C_{l,q}}{(1-r)^{2q+1}}\, I,$$

where $C_{l,q} = C_l C_q \chi_l$, which was to be proved.

In the case $N > 3$ the proof is analogous.

§ 3. For definiteness, let us take $N = 3$; however, the method used is general for any N. Consider an arbitrary bounded three-dimensional domain G with boundary Γ of class $C^{(2)}$.

Taking as a basis the results of the work of S. M. Nikol' skii ([3]), near each point $x_0 = (x_1^0, x_2^0, x_3^0) \in \Gamma$ we construct a neighborhood $\Delta = \Delta(\bar{x}_0)$ which cuts out on Γ a piece σ defined by an equation explicitly expressed through one of the coordinates. For definiteness let this equation be

$$x_3 = \varphi(x_1, x_2), \qquad \Delta = \{a < x_1 < b;\ c < x_2 < d\}, \qquad \varphi \in C^{(2)}(\bar{\Delta}). \quad (3,1)$$

Put

$$u_1 = x_1 + h\alpha_1, \qquad u_2 = x_2 + h\alpha_2, \qquad u_3 = \varphi(x_1, x_2) + h\alpha_3, \quad (3,2)$$

where $h > 0$; $\alpha_i = \alpha_i(x_1, x_2)$ are the direction cosines of the angles formed by the inner normal to Γ with the axes x_i. For sufficiently small $\delta_0 > 0$, the equalities (3,2) give a one-to-one and continuously differentiable transformation of the points (x_1, x_2, h) $((x_1, x_2) \in \Delta,\ 0 < h < \delta_0)$ into the points $(u_1, u_2, u_3) \in U_{\Delta,\delta_0} \subset G$. Define also the domain

$$U_{\Delta',\,\delta_0/2} \subset U_{\Delta,\delta_0}, \qquad \Delta' = \left\{a + \frac{\delta_0}{2} < x_1 < b - \frac{\delta_0}{2};\ c + \frac{\delta_0}{2} < x_2 < d - \frac{\delta_0}{2}\right\}.$$

Let

$$x_i = \psi_i(u_1, u_2, u_3), \qquad h = \psi_3(u_1, u_2, u_3) \quad (i = 1, 2) \quad (3,3)$$

be the transformations inverse to (3,2), and

$$\left|\frac{\partial \psi_k}{\partial u_j}\right| \leq M \qquad (k, j = 1, 2, 3), \qquad (u_1, u_2, u_3) \in U_{\Delta,\delta_0}.$$

Then, according to ([3]), any ball ω with center $Q(x_1^0, x_2^0, \delta_0/2) \in U_{\Delta',\delta_0/2}$ and radius $\mu\delta_0/2$ belongs to the domain U_{Δ,δ_0}, i.e. does not go beyond the boundary Γ of the domain G, provided only that $0 < \mu \leq 1/3M$. By the properties of the function $\varphi(x_1, x_2)$, $|\partial\varphi/\partial l| \leq L < +\infty$, where l is any direction in the plane (x_1, x_2). Take $T = \max(L, M)$.

Construct on Δ' a grid of squares with side $h = \delta_0/96(1+T)^3$, and a grid of concentric balls V and V_1 of radii $\rho = 7\delta_0/48(1+T)$ and $\rho_1 = \delta_0/8(1+T)$, respectively, with centers lying on the surface $\Gamma_{\delta_0/2}$ (Γ_η is the surface at distance η from Γ along the inner normals) and projecting in the direction h to the nodes of the grid on Δ'.

On the basis of the Heine–Borel lemma, the surface Γ can be covered by a finite number of the neighborhoods $\Lambda(\bar{x}_0)$ defined above, and in each such neighborhood one can construct the above-described net of balls V and V_1.

Obviously, there exists a closed (geometrically) piecewise-smooth surface $\gamma_\delta \subset G$ satisfying conditions (1,5) and (1,6). In addition, in the present paper it has been shown that the balls V_1 (of smaller radius) completely contain the volume layer determined by the inequality

$$\frac{\delta_0}{2} - \frac{\delta_0}{16(1+T)} \leq h \leq \frac{\delta_0}{2},$$

whence inequality (1,7) follows. Let us also note that each ball V of the construction intersects only a finite number of balls.

Using inequality (1.4), we obtain

$$\iint_{\gamma_\delta} u^{(q)2}\, d\gamma_\delta < \sum_{\nu=1}^{n} \iint_{C_{r_\nu}} u^{(q)2}\, dS < \frac{C_{l,q}}{\delta^{2q+1}} \sum_{\nu=1}^{n} \iiint_{V_\nu} u^2\, dV_\nu < \frac{C_{l,q}^*}{\delta^{2q+1}} \iiint_G u^2\, dG,$$

which is what was required to prove. Here $C_{l,q}^* = mC_{l,q}$, where m is the greatest number of balls V of the net that intersect any fixed ball of the net; the number m depends on G, but not on δ_0; n is the total number of balls V of the net, ν is the serial number of the ball V_ν; C_{r_ν} is the full surface of the ball $V_{1\nu}$.

The estimates obtained, (1,4) and (1,8), may be used in the solution of certain boundary-value problems, for example, in proving uniqueness of the solution of the first boundary-value problem for a polyharmonic equation.

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Note: Figure translations are in progress. See original paper for figures.