Academician of the Academy of Sciences of the Armenian SSR M. M. Dzhrbashyan

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Abstract

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MATHEMATICS

Academician of the Academy of Sciences of the Armenian SSR M. M. Dzhrbashyan

EXPANSIONS IN SYSTEMS OF RATIONAL FUNCTIONS WITH FIXED POLES

In the author' s paper (1) a special system of rational fractions $\{M_k(z)\}$ $(k = 0, 1, 2, ...)$ with fixed poles $\{\omega_k\}$ $(k = 0, 1, 2, ...)$, lying outside a given simply connected domain $G^{(+)}$ with a closed rectifiable Jordan boundary Γ, was constructed. This system was a natural generalization of the Faber polynomial system for the case when all poles are not concentrated at the point $z = \infty$, but lie on a given sequence of points $\{\omega_k\}$.

If the function

$$w = \Phi(z) \quad (z = \psi(w)), \tag{1}$$

subject to the normalization condition $\Phi(\infty) = \infty,\ \Phi'(\infty) > 0$, maps conformally the domain $G^{(-)}$ complementary to $G^{(+)}$ onto the domain $|w| > 1$, then, under a certain restriction imposed on the contour Γ* and on the density of the distribution of the poles $\{\omega_k\}$, consisting in the requirement

$$\sum_{k=0}^{\infty} \left(1 - |\Phi(\omega_k)|^{-1}\right) = \infty, \tag{2}$$

the possibility was established of a uniformly convergent expansion inside the domain $G^{(+)}$

$$f(z) = \sum_{k=0}^{\infty} c_k M_k(z) \tag{3}$$

for classes of functions representable in $G^{(+)}$ by an integral of Cauchy type.

Subsequently G. Ts. Tumarkin (2) showed that, with the aid of a device usually used in the theory of Faber polynomials, the system $\{M_k(z)\}$ can be modified somewhat so as to obtain the expansion without additional restrictions on the contour Γ. At the same time he established that condition (2) is not only sufficient but also necessary; moreover, when the series (2) converges, a description was given of the class of those functions which nevertheless admit an expansion of the form (3).

In the present note we give formulations of several new and more general results on expansions in systems of rational fractions with fixed poles.

1°. Let K be a bounded continuum containing more than one point, and let $G^{(-)}$ be one of the domains adjacent to it which contains the point $z = \infty$. Let further $\{\omega_k\} \subset G^{(-)}$ $(k = 0, 1, 2, \ldots)$ be an arbitrary sequence complex numbers (among which there may also be numbers equal to ∞, as well as numbers of finite or even infinite multiplicity). Let us introduce the orthonormal system on the circle $|w| = 1$ of Malmquist rational functions (^{3})

$$\varphi_0(w) = \frac{(1 - |\alpha_0|^2)^{1/2}}{1 - \overline{\alpha}_0 w},$$

$$\varphi_n(w) = \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \overline{\alpha}_n w} \prod_{k=0}^{n-1} \frac{\alpha_k - w}{1 - \overline{\alpha}_k w} \frac{|\alpha_k|}{\alpha_k} \qquad (n = 1, 2, \ldots), \tag{4}$$

where $\alpha_k = [\overline{\Phi(\omega_k)}]^{-1}$ $(k = 0, 1, 2, \ldots)$, and we assume that $|\alpha_k|/\alpha_k = -1$ when $\alpha_k = 0$.

The function $\varphi_n[\Phi(z)]|\Phi'(z)|^s$ $(0 \leq s \leq 1)$, where $w = \Phi(z)$ $(\Phi(\infty) = \infty,\ \Phi'(\infty) > 0)$ maps the domain $G^{(-)}$ onto $|w| > 1$, is holomorphic everywhere in the domain $G^{(-)}$ except at the points $z = \omega_0, \ldots, \omega_n$, where it has poles, the multiplicity of each of which is equal to the multiplicity of its occurrence in the group of numbers $\{\omega_0, \ldots, \omega_n\}$. Denote by $M_n^{(s)}(z)$ the sum of the principal parts together with the constant terms of the expansion of the function $\varphi_n[\Phi(z)]|\Phi'(z)|^s$ in neighborhoods of its distinct singular points $\omega_0, \omega_1, \ldots, \omega_n$.

We shall call the system of rational functions $\{M_n^{(s)}(z)\}$ $(n = 0, 1, 2, \ldots)$ the **system generated by the continuum K and the sequence $\{\omega_k\}$**.

It is easy to see that in the case when $\omega_0 = \omega_1 = \cdots = \infty$, $\varphi_n[\Phi(z)] = [\Phi(z)]^n$ $(n = 0, 1, 2, \ldots)$, whence the system $\{M_n^{(s)}(z)\}$ reduces to the known system of

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* This restriction consisted in the requirement that the limit

$$\lim_{\rho \to 1+0} \int_0^{2\pi} \left|\psi'(\rho e^{i\theta})\right|^p d\theta$$

be finite for $p = 2$, whereas for an arbitrary rectifiable Jordan curve Γ one can only assert the finiteness of this limit for $p = 1$.

Faber polynomials (for $s = 0$) and its modifications (for $0 < s \leq 1$). Like the Faber polynomials, the functions $M_n^{(s)}(z)$ admit an integral representation.

Denote by Γ_ρ ($\rho > 1$) the curve $|\Phi(z)| = \rho$, and by $G_\rho^{(-)}$ and $G_\rho^{(+)}$ its exterior and interior domains, respectively. Then the following is true:

Lemma 1. *Let* $1 < \rho < R_n = \min_{0 \leq k \leq n}\{|\Phi(\omega_k)|\}$*; then:*

1) *For* $z \in G_\rho^{(+)}$*, in particular for* $z \in K$*,*

$$M_k^{(s)}(z) = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\varphi_k[\Phi(\xi)]\,|\Phi'(\xi)|^s}{\xi - z}\,d\xi \qquad (k = 0, 1, \dots, n). \tag{5}$$

2) *For* $z \in G_\rho^{(-)}$

$$M_k^{(s)}(z) = \varphi_k[\Phi(z)]|\Phi'(z)|^s + \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\varphi_k[\Phi(\xi)]\,|\Phi'(\xi)|^s}{\xi - z}\,d\xi \qquad (k = 0, 1, \dots, n). \tag{6}$$

3) *Moreover, if* $K = \overline{G^{(+)}}$*, where* $G^{(+)}$ *is a simply connected domain with closed rectifiable Jordan boundary* Γ*, then formulas (5)–(6) hold for all* n*, if the contour of integration* Γ_ρ *is replaced by* Γ*.*

2°. Let the boundary of the mutually complementary domains $G^{(+)}$ and $G^{(-)}(\ni \infty)$ be a closed rectifiable Jordan curve Γ. We shall say that $\Gamma \subset U_s$ $(0 \leq s \leq 1)$, if the function $z = \psi(w)$ $(\psi(\infty) = \infty)$, mapping the domain $|w| > 1$ onto the domain $G^{(-)}$, satisfies the condition

$$\int_0^{2\pi} |\psi'(e^{i\vartheta})|^{2(1-s)}\,d\vartheta < +\infty. \tag{7}$$

We note that always $\Gamma \subset U_{1/2}$, whereas $U_s \subset U_{1/2}$ for $0 \leq s < \frac{1}{2}$, and $U_s \equiv U_{1/2}$ for $\frac{1}{2} \leq s \leq 1$.

The main theorem on expansions in the systems $\{M_k^{(s)}(z)\}$ is based on the following lemma.

Lemma 2. Let $\Gamma \subset U_s$; then:

1) If $z \in G^{(+)}$, then under the condition

$$\sum_{k=0}^{\infty} \left(1 - |\Phi(\omega_k)|^{-1}\right) = +\infty \tag{8}$$

the expansion

$$\frac{[\psi'(w)]^{1-s}}{\psi(w)-z} = \sum_{k=0}^{\infty} M_k^{(s)}(z) \left\{ \frac{1}{w} \varphi_k\left(\frac{1}{w}\right) \right\}' \qquad (|w|>1), \tag{9}$$

is valid, converging absolutely and uniformly inside the domain $|w| > 1$ and in the mean on its boundary.

2) If $z \in G^{(+)}$, then under the condition

$$\sum_{k=0}^{\infty} \left(1 - |\Phi(\omega_k)|^{-1}\right) < +\infty \tag{10}$$

the representation

$$\frac{[\psi'(w)]^{1-s}}{\psi(w)-z} = \sum_{k=0}^{\infty} M_k^{(s)}(z) \left\{ \frac{1}{w} \varphi_k\left(\frac{1}{w}\right) \right\} + \frac{\omega\left(\frac{1}{w}; z\right)}{wB(w)} \qquad (|w|>1), \tag{11}$$

is valid, where the series on the right has the same character of convergence, and

$$B(w) = \prod_{k=0}^{\infty} \frac{1-\Phi(\omega_k)w}{\Phi(\omega_k)-w} \frac{|\Phi(\omega_k)|}{\Phi(\omega_k)}, \tag{12}$$

while the function

$$\omega(\zeta; z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{B(t)[\psi'(t)]^{1-s}}{[\psi(t)-z](1-t\zeta)}\, dt \qquad (|\zeta|<1) \tag{13}$$

belongs to the Riesz class H_2.

Denote by $K_2^{(s)}(G^{(+)})$ the class of functions holomorphic in the domain $G^{(+)}$ and representable in the form of a Cauchy-type integral

$$f(z) = K(z; g) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\xi)}{\xi - z}\, d\xi, \qquad z \in G^{(+)}, \tag{14}$$

where

$$\int_{\Gamma} \left|g(\xi)\, [\Phi'(\xi)]^{1/2-s}\right|^2 |d\xi| < +\infty. \tag{15}$$

Theorem 1. Let $\Gamma \subset U_s$ and

$$f(z) = K(z; g) \in K_2^{(s)}(G^{(+)}).$$

Then:

1) If

$$\sum_{k=0}^{\infty} \left(1 - |\Phi(\omega_k)|^{-1}\right) = +\infty,$$

then the expansion

$$f(z) = \sum_{k=0}^{\infty} c_k(g)\, M_k^{(s)}(z), \qquad z \in G^{(+)}, \tag{16}$$

is valid, converging absolutely and uniformly inside the domain $G^{(+)}$.

2) If

$$\sum_{k=0}^{\infty} \left(1 - |\Phi(\omega_k)|^{-1}\right) < +\infty,$$

then the representation holds

$$f(z) = \sum_{k=0}^{\infty} c_k(g) M_k^{(s)}(z) + \frac{1}{2\pi i} \int_\Gamma \frac{g(\xi)[\Phi'(\xi)]^{1-s}}{\Phi(\xi) B[\Phi(\xi)]} \omega\left(\frac{1}{\Phi(\xi)}; z\right) d\xi, \qquad z \in G^{(+)}, \tag{17}$$

where the series on the right has the same character of convergence and, moreover, converges everywhere in the domain $G^{(-)}$ except for the points $\{\omega_k\}$ $(k = 0, 1, 2, ...)$.

In formulas (16) and (17)

$$c_k(g) = \frac{1}{2\pi i} \int_\Gamma g(\xi) \frac{[\Phi'(\xi)]^{1-s}}{\Phi(\xi)} \overline{\varphi_k[\Phi(\xi)]}\, d\xi \qquad (k = 0, 1, 2, ...), \tag{18}$$

and

$$\sum_{k=0}^{\infty} |c_k(g)|^2 < +\infty.$$

Thus, formula (17) of this theorem may be understood as an effective completion of the system $\{M_k^{(s)}(z)\}$ in the case when, owing to condition (10), it is not complete in the sense of assertion 1) of the theorem.

3°. Under condition (10), let us assign to the class $E_2^{(s)}(G^{(-)};\omega_k)$ the set of functions $F(z)$, meromorphic in the domain $G^{(-)}$, which are representable in the form

$$F(z)=\frac{B[\Phi(z)]}{\Phi(z)}[\Phi'(z)]^s\widetilde{F}\left(\frac{1}{\Phi(z)}\right),\qquad z\in G^{(-)},\tag{19}$$

where $\widetilde{F}(w)\in H_2$ is arbitrary. We shall further denote by $\mathfrak{H}_2\{G^{(+)},G^{(-)};\omega_k\}$ the class of functions $f(z)$, defined on the set $G\{\omega_k\}=G^{(+)}+G^{(-)}-\{\omega_k\}^*$ and satisfying the conditions

$$\begin{aligned} &\text{a)}\quad f(z)=K(z;g)\in K_2^{(s)}(G^{(+)}), && z\in G^{(+)},\\ &\text{b)}\quad f(z)=K(z;g)+F(z), && z\in G^{(-)},\end{aligned}\tag{20}$$

where $F(z)\in E_2^{(s)}(G^{(-)};\omega_k)$ and $g(\xi)=F(\xi)$ for almost all points $\xi\in\Gamma$. Let us note that from the representations (20) it follows that, for almost all points $\xi_0\in\Gamma$,

$$\lim_{\varepsilon\to 0}\{f(\xi_0+i\varepsilon e^{i(\varphi_0+\psi_0)})-f(\xi_0-i\varepsilon e^{i(\varphi_0+\psi_0)})\}=0,$$

where φ_0 is the angle of inclination of the tangent to the curve Γ at the point ξ_0, and ψ_0 ($|\psi_0|<\pi/2$) is arbitrary. Thus, the functions of the class $\mathfrak{H}_2\{G^{(+)},G^{(-)};\omega_k\}$, in a certain sense, are monogenic functions in the sense of Borel on the entire z-plane.

From Lemma 1 and Theorem 1 it follows:

Theorem 2. The class $\mathfrak{H}_2\{G^{(+)},G^{(-)};\omega_k\}$ coincides with the set of functions representable in the form of the series

$$f(z)=\sum_{k=0}^{\infty}c_kM_k^{(s)}(z),\qquad z\in G\{\omega_k\},\tag{21}$$

where $\{c_k\}$ ($k=0,1,2,\dots$) is an arbitrary sequence of complex numbers for which

$$\sum_{k=0}^{\infty}|c_k|^2<\infty.$$

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

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* That is, the entire z-plane except for the points of the curve Γ and the points of the sequence $\{\omega_k\}$ $(k = 0, 1, 2, ...)$.

Note: Figure translations are in progress. See original paper for figures.