V. S. KOROLYUK

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. S. KOROLYUK

ON THE ASYMPTOTICS OF DISTRIBUTIONS OF MAXIMAL DEVIATIONS

(Presented by Academician A. N. Kolmogorov, 28 VIII 1961)

1. In the present note, the algorithm set forth in [1] for constructing asymptotic expansions for the distributions of maximal deviations of normalized sums is formulated in an improved form

$$\eta_k = \varepsilon \sum_{r=1}^{k} \xi_r \qquad \left(0 \le k \le n;\ \eta_0 = 0;\ \varepsilon = \frac{1}{\sqrt{n}}\right) \tag{1}$$

of independent identically distributed random variables ξ_r $(r \ge 1)$ with distribution function $P(x)$, either absolutely continuous or lattice, and with finite moments of order $N+5$, moreover

$$\mathbf{M}\xi_r = 0; \qquad \mathbf{D}\xi_r = 1; \qquad \mathbf{M}\xi_r^q = \alpha_q \quad (q \ge 3).$$

Denote

$$p_\varepsilon(k,x) = \begin{cases} \varepsilon^{-1}\mathsf{P}\{\eta_k = x\}, & \text{in the lattice case,} \\ \dfrac{d}{dx}\mathsf{P}\{\eta_k < x\}, & \text{in the continuous case.} \end{cases}$$

Let $z_-(t)$ and $z_+(t)$ be sufficiently smooth functions on the interval $0 \le t \le 1$, with $z_-(0) < 0 < z_+(0)$; $z_-(t) < z_+(t)$. Introduce the joint distribution of the sums (1), local with respect to η_n:

$$B_\varepsilon(z_-, z_+; x) = \varepsilon^{-1}\mathsf{P}\left\{z_-\left(\frac{k}{n}\right) < \eta_k < z_+\left(\frac{k}{n}\right),\ 0 \le k \le n;\ \eta_n = x\right\}$$

in the lattice case, or

$$B_\varepsilon(z_-, z_+; x) = \frac{d}{dx}\mathsf{P}\left\{z_-\left(\frac{k}{n}\right) < \eta_k < z_+\left(\frac{k}{n}\right),\ 0 \le k \le n;\ \eta_n \le x\right\}$$

---

[1] As printed on the visible page.

in the continuous case.

The starting point in the algorithm is the boundary-value problem for an integral equation with small parameter ε:

$$\mathbf{P}_\varepsilon u_\varepsilon(k,x) \equiv \int u_\varepsilon(k, x-\varepsilon y)\, dP(y) - u_\varepsilon(k+1,x) = 0. \tag{2}$$

The required function $u_\varepsilon(k,x)$ is determined by equation (2) for

$$z_-\left(\frac{k}{n}\right) < x < z_+\left(\frac{k}{n}\right)$$

under the conditions

$$u_\varepsilon(0,x) = 0, \tag{3}$$

$$u_\varepsilon(k,x) = p_\varepsilon(k,x), \qquad x \notin \left(z_-\left(\frac{k}{n}\right), z_+\left(\frac{k}{n}\right)\right) \quad (1 \le k \le n).$$

Let us note that for $z_-\left(\frac{k}{n}\right) < x < z_+\left(\frac{k}{n}\right)$

$$B_\varepsilon(z_-, z_+; x) = p_\varepsilon(n,x) - u_\varepsilon(n,x). \tag{4}$$

The last relation explains the choice of the boundary-value problem for equation (2).

2. Suppose that a refinement of the local limit theorem holds (see, for example, (2), Ch. 8), i.e., for sufficiently large k

$$p_\varepsilon(k,x) = p_0(t,x) + \sum_{r=0}^{N+2} \varepsilon^r p_r(t,x) + O(\varepsilon^{N-3}), \tag{5}$$

where

$$p_0(t,x) = \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2t}, \qquad p_r(t,x) = t^{-r/2} P_r(-\varphi(\lambda)),$$

$$\varphi(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2}, \qquad \lambda = \frac{x}{\sqrt{t}}, \qquad t = \frac{k}{n}$$

(the definition of $P_r(-\varphi)$, see (2), Ch. 8).

Then, in accordance with the general scheme of asymptotic analysis set forth in (1), we have the following.

Theorem. *Under the conditions of existence of the expansion (5) and under sufficient smoothness of the boundary curves $z_-(t)$, $z_+(t)$, for the solution $u_\varepsilon(k,x)$ of equation (2) under conditions (3) there is the asymptotic representation*

$$u_\varepsilon(k,x) = \sum_{r=0}^{N} \varepsilon^r u_r(t,x) +$$

$$+ \sum_{r=0}^{N+1} \varepsilon^{r+1} \left[V_r^- \left(t, \frac{x - z_-}{\varepsilon} \right) + V_r^+ \left(t, \frac{z_+ - x}{\varepsilon} \right) \right] + O(\varepsilon^N), \tag{6}$$

in which the regular terms of the asymptotics $u_r(t,x)$ are determined from the differential equations

$$\mathbf{L}_0 u_0 \equiv \frac{1}{2} \frac{\partial^2 u_0}{\partial x^2} - \frac{\partial u_0}{\partial t} = 0; \tag{7}$$

$$\mathbf{L}_0 u_r = -\sum_{m=0}^{r-1} \mathbf{L}_{r-m} u_m \qquad (m = 1, 2, \dots, N) \tag{8}$$

with the conditions

$$u_r(0,x) = 0, \qquad z_-(0) < x < z_+(0) \qquad (0 \le r \le N); \tag{9}$$

$$u_0(t, z_\pm(t)) = p_0(t, z_\pm(t)); \tag{10}$$

$$u_r(t, z_\pm(t)) = p_r(t, z_\pm(t)) - V_{r-1}^\pm(t,0) \qquad (1 \le r \le N), \tag{11}$$

and the boundary layers $V_r^\pm(t,s)$ are determined as solutions, decreasing to zero as $s \to \infty$, of integral equations on the half-line $s \ge 0$

$$\mathbf{P}_0^\pm V_0^\pm \equiv \int V_0^\pm(t, s \pm y)\, dP(y) - V_0^\pm(t,s) = 0; \tag{12}$$

$$\mathbf{P}_0^\pm V_r^\pm = \sum_{m=1}^{r} \mathbf{P}_m V_{r-m}^\pm \qquad (r = 1, 2, \dots, N+1). \tag{13}$$

with given values for $s < 0$:

$$V_r^{\pm}(t,s) = V_r^{\pm}(t,0) - \sum_{\nu=1}^{r+1} \frac{(\mp s)^{\nu}}{\nu!} \frac{\partial^{\nu}}{\partial x^{\nu}} \left[u_{r+1-\nu}(t,x) - p_{r+1-\nu}(t,x)\right]_{x=z_{\pm}} . \qquad (14)$$

Here

$$\mathbf{L}_{2k-1} u \equiv -\frac{\alpha_{2k+1}}{(2k+1)!} \frac{\partial^{2k+1} u}{\partial x^{2k+1}}; \qquad \mathbf{L}_{2k} u \equiv \frac{\alpha_{2k}}{(2k)!} \frac{\partial^{2k} u}{\partial x^{2k}} - \frac{1}{k!} \frac{\partial^{k} u}{\partial t^{k}} \quad (k \geq 1); \quad (15)$$

$$\mathbf{P}_{2k} V \equiv \frac{1}{k!} \frac{\partial^{k} V}{\partial t^{k}}, \qquad \mathbf{P}_{2k+1} V \equiv 0 \quad (k \geq 1). \qquad (16)$$

3. The proof of the theorem is carried out by the method of upper and lower functions in the same spirit as in the paper (1). At the same time, we note that the solution $u_0(t,x)$ of the heat equation (7) with conditions (9) and (10) is a sufficiently smooth function in the domain $Q\{0 \leq t \leq 1,\ z_-(t) < x < z_+(t)\}$, since

$$u_0(t,x) = p_0(t,x) - p_Q(t,x;0,0), \qquad (17)$$

where $p_Q(t,x;\tau,y)$ is the Green function of equation (1) in the domain Q, and $p_0(t,x)$ is the fundamental solution of equation (1). This ensures the smoothness of the remaining regular terms of the asymptotics.

4. For solutions of equations (8) an integral representation with the Green function $p_Q(t,x;\tau,y)$ can be given. The solution of equations (12) and (13) for the boundary layers is set out in (1).

From the theorem, taking (4) and (5) into account, we obtain an asymptotic expansion for the distribution $B_\varepsilon(z_-,z_+;x)$. In particular, one can compute in elementary functions the asymptotic terms for the distribution of maximal deviations of sums (1), i.e. for the joint distribution of the random variables

$$\overline{\eta}_n = \max_{0 \leq k \leq n} \eta_k, \qquad \underline{\eta}_n = \min_{0 \leq k \leq n} \eta_k, \eta_n.$$

For $z_-(t) = z_- = \text{const}$ and $z_+(t) = z_+ = \text{const}$, the distribution $B_\varepsilon(z_-,z_+;x)$ for every x $(z_- < x < z_+)$ is asymptotically representable in the form

$$B_\varepsilon(z_-,z_+;x) = B_0(z_-,z_+;x) + \varepsilon B_1(z_-,z_+;x) + O(\varepsilon^2),$$

where

$$B_0(z_-, z_+; x) = \sum_{k=-\infty}^{\infty} (-1)^k \varphi(x - 2z_k);$$

$$B_1(z_-, z_+; x) = \sum_{k=-\infty}^{\infty} \left[\left(\frac{\alpha_3}{3} + (-1)^k 2\rho_k \right) \frac{d}{dx} \varphi(x - 2z_k) + (-1)^k \frac{\alpha_3}{6}\, x \,\frac{d^2}{dx^2} \varphi(x - 2z_k) \right].$$

Here $\varphi(n)$ is the density of the normal distribution with zero mathematical expectation and unit variance;

$$z_{2k} = k(z_+ - z_-); \qquad z_{2k-1} = z_{2k} + z_-;$$

$$\rho_{2k} = k(\rho_+ - \rho_-); \qquad \rho_{2k-1} = \rho_{2k} + \rho_-;$$

ρ_+ (ρ_-) is the mathematical expectation of the limiting value of the first overshoot across the level X as $X \to +\infty$ ($X \to -\infty$) of the sequence of sums

$$\sum_{r=1}^{k} \xi_r \quad (r \geq 1).$$

From the relation

$$\frac{B_\varepsilon(z_-, z_+; x)}{p_\varepsilon(n, x)} = \mathbb{P}\left\{ z_-\left(\frac{k}{n}\right) < \eta_k < z_+\left(\frac{k}{n}\right),\ 0 \leq k \leq n \,\middle|\, \eta_n = x \right\}$$

it is easy to obtain an asymptotic expansion for the conditional (with respect to η_n) joint distribution of the sums (1). An asymptotic representation for the distribution

$$B_\varepsilon(z_-, z_+) = \mathbb{P}\left\{ z_-\left(\frac{k}{n}\right) < \eta_k < z_+\left(\frac{k}{n}\right),\ 0 \leq k \leq n \right\}$$

is constructed from the relations

$$B_\varepsilon(z_-, z_+) = \begin{cases} \displaystyle\int_{z_-(1)}^{z_+(1)} B_\varepsilon(z_-, z_+; x)\, dx, & \text{in the continuous case,} \\ \displaystyle\sum_{z_-(1) < x < z_+(1)} \varepsilon B_\varepsilon(z_-, z_+; x), & \text{in the lattice case.} \end{cases}$$

In conclusion, we note that the algorithm formulated in the theorem is also applicable in the case where the original distribution of the summands depends on ε in a sufficiently smooth manner, i.e., when considering a sequence of series of independent random variables identically distributed within each series. In this case only the form of the operators L_r and $\hat{P}_r$ $(r \geq 1)$ will change.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
1 VII 1961

References

1. V. S. Korolyuk, *Probability Theory and Its Applications*, 6, issue 3 (1961).
2. B. V. Gnedenko, A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Moscow–Leningrad, 1949.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.