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Abstract

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ON IRREDUNDANT DISJUNCTIVE NORMAL FORMS FOR SOME CLASSES OF FUNCTIONS OF THE ALGEBRA OF LOGIC

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1. This note concerns the theory of disjunctive normal forms (d.n.f.) $^{(1)}$. The note studies the relation between the complexity of an arbitrary irredundant d.n.f. and a minimal d.n.f. of a function f, taking into account the dimension of the function f. The dimension of a function $f(x_1, \dots, x_n)$ (denoted by $\operatorname{Dim} f$) is defined as the maximum of the dimensions of those subcubes of the n-dimensional unit cube on which $f(x_1, \dots, x_n) = 1$.

The principal problem in the theory of d.n.f. is the finding of minimal (with respect to the number of conjunctions, and to the number of letters) d.n.f.' s. Since algorithms for solving this problem require a large exhaustive search $^{(2)}$, an algorithm for simplifying d.n.f.' s is widely used, consisting in obtaining some irredundant d.n.f. of the function f. In order to estimate how many times the complexity of an irredundant d.n.f. can exceed the complexity of a minimal d.n.f., let us associate with the function f the number $Y(f) = \max[I(T_1)/I(T_2)]$, where the maximum is taken over all pairs of irredundant d.n.f.' s T_1, T_2 of the function f; $I(T_1), I(T_2)$ are the numbers of conjunctions in these d.n.f.' s. From cardinality considerations one can establish that $Y(f) \leqslant 2^{\operatorname{Dim} f}$. At the same time, $Y(f) = 2^{\operatorname{Dim} f}$ occurs only for functions for which $\operatorname{Dim} f = 0$.

In the present note it is clarified how sharp the estimate $Y(f) \leqslant 2^{\operatorname{Dim} f}$ is. Namely, functions $f_n(x_1, \dots, x_n)$ are constructed for which $Y(f_n) \to 2^{\operatorname{Dim} f_n}$ as $n \to \infty$, and $\operatorname{Dim} f_n$ grows with n; moreover, $\operatorname{Dim} f_n$ may be sufficiently close to n, for example $\operatorname{Dim} f_n > n - [n/k]$, where $k \geqslant 2$ and does not depend on n. This means that, if we take at random some irredundant d.n.f. instead of a minimal d.n.f. for a given function f, we may obtain a d.n.f. almost $2^{\operatorname{Dim} f}$ times more complex than the minimal d.n.f. In $^{(3)}$, Theorem 3, functions were studied whose dimension is less than half the number of variables. However, functions whose dimension is close to the number of variables are of greatest interest. It is precisely for such functions that the replacement of a minimal d.n.f. by a randomly chosen irredundant d.n.f. will be least justified.

2. Let $D(n)$ be an integer-valued function, $0 \leqslant D(n) \leqslant n$; let $\mathfrak{M}_{D(n)}$ be the class of functions $f(x_1, \dots, x_n)$ for which $\operatorname{Dim} f \leqslant D(n)$; $Y(\mathfrak{M}_{D(n)}) =$

$$\max_{f\in\mathfrak{M}_{D(n)}} Y(f).$$

Theorem. For any integer-valued function $\mathcal{D}(n)$, $0 \leqslant \mathcal{D}(n) \leqslant n$, one can specify such a function $D(n)$ and a class $\mathfrak{M}_{D(n)}$ that $D(n) \sim \mathcal{D}(n)$,

$$Y(\mathfrak{M}_{D(n)}) \sim 2^{D(n)}.$$

The proof consists in constructing, for any $\delta > 0$ and any function $\mathcal{D}(n)$, $0 \leqslant \mathcal{D}(n) \leqslant n$, such a function $f_n(x_1, \dots, x_n)$ that, for $n \geqslant n_\delta$,

$$1 - \delta \leqslant \operatorname{Dim} f_n / \mathcal{D}(n) \leqslant 1 + \delta, \qquad Y(f_n) \geqslant 2^{\operatorname{Dim} f_n - \delta}.$$

(Then, as $D(n)$, we may take $\operatorname{Dim} f_n$.)

The construction of the function $f_n(x_1, \dots, x_n)$ will be carried out in three stages. First, Lemma 1 on "superpositions" of irredundant d.n.f.' s will be formulated (Sec. 3). Then (Sec. 4), relying on Lemma 1, we shall consider Lemma 2 on powers

support functions (definition see below), which reduces the construction of the function $f(x_1, \dots, x_n)$ to a simpler problem—the construction of support functions (item 5).

3. Denote by $\mathbf{x}^n$ the set of variables $x_1, \dots, x_n$, and by $\mathbf{x}_1^{n_1}, \mathbf{x}_2^{n_2}, \dots$ the sets of variables $x_{11}, \dots, x_{1n_1}; x_{21}, \dots, x_{2n_2}, \dots$. Let Φ be a formula in the basis $\vee, \&, ^{-}$ with parentheses. The DNF obtained from the formula Φ after formal expansion of all parentheses will be called the DNF of the formula Φ.

Lemma 1. *Suppose there are given a function $f(\mathbf{u}^s)$, functions $g_1(\mathbf{v}_1^{t_1}), \bar{g}_1(\mathbf{v}_1^{t_1}), \dots, \dots, g_k(\mathbf{v}_k^{t_k}), \bar{g}_k(\mathbf{v}_k^{t_k})$ and, respectively, their irredundant DNFs $T(\mathbf{u}^s)$ and $S_{10}(\mathbf{v}_1^{t_1}), S_{11}(\mathbf{v}_1^{t_1}), \dots, S_{k0}(\mathbf{v}_k^{t_k}), S_{k1}(\mathbf{v}_k^{t_k}), 1 \leq k \leq s$. Denote by $T(\mathbf{u}^{s-k}, S_1(\mathbf{v}_1^{t_1}), \dots, \dots, S_k(\mathbf{v}_k^{t_k}))$ the DNF of the formula obtained by substituting in the DNF $T(\mathbf{u}^s)$ the DNFs $S_{i\alpha}(\mathbf{v}_i^{t_i})$ in place of all occurrences of u_{s-k+i}^{α} in $T(\mathbf{u}^s)$, $\alpha = 0, 1,\ i = 1, \dots, k$. Assume that all variables $u_{s-k+1}, \dots, u_s$ are distinct and that functions substituted in place of distinct variables themselves have no variables in common. Then the DNF $T(\mathbf{u}^{s-k}, S_1(\mathbf{v}_1^{t_1}), \dots, S_k(\mathbf{v}_k^{t_k}))$ will be an irredundant DNF of the function $f(\mathbf{u}^{s-k}, g_1(\mathbf{v}_1^{t_1}), \dots, g_k(\mathbf{v}_k^{t_k}))$.*

4. An irredundant DNF $T(x, y, \mathbf{v}^k)$, $k \geq 2$, will be called **linear in the variables** x, y if each conjunction of the DNF T contains one and only one of the variables x, y. An irredundant DNF $T(x, y, \mathbf{v}^k)$ will be called **non-linear in the variables** x, y if each conjunction of the DNF T contains both variables x, y.

A function $f(x, y, \mathbf{v}^k)$, $k \geq 2$, will be called a **support function** in x, y if: a) $\bar{f}(x, y, \mathbf{v}^k) = f(\bar{x}, \bar{y}, \mathbf{v}^k)$; b) the function $f(x, y, \mathbf{v}^k)$ has an irredundant DNF

$\mathfrak{A}(x, y, \mathbf{v}^k)$, linear in the variables x, y, with $l(\mathfrak{A}) = 2^k$, $\operatorname{Dim}\mathfrak{A} = 1$; c) the function $f(x, y, \mathbf{v}^k)$ has an irredundant DNF $\mathfrak{B}(x, y, \mathbf{v}^k)$, nonlinear in the variables x, y, with $l(\mathfrak{B}) \geq 2^{k+1-\varepsilon(k)}$, where $\varepsilon(k) \to 0$ as $k \to \infty$, $\operatorname{Dim}\mathfrak{B} = 1$.

For an arbitrary support function $f(x, y, \mathbf{v}^k)$ and an arbitrary number of variables s, $s \geq 1$, call the function $L(\mathbf{w}^s) = w_1 + \cdots + w_s \pmod 2$ the 0-th degree of the function $f(x, y, \mathbf{v}^k)$ and denote it by $f^0_{k,s}$. If the $(r-1)$-st degree of the function $f(x, y, \mathbf{v}^k)$ has already been defined and denoted by $f^{r-1}_{k,s}(\mathbf{w})$, then call the function $f(f^{r-1}_{k,s}(\mathbf{x}), f^{r-1}_{k,s}(\mathbf{y}), \mathbf{v}^k)$ the r-th degree of the function $f(x, y, \mathbf{v}^k)$ and denote it by $f^r_{k,s}$. Denote by $n_{r,k,s}$ the number of variables of the function $f^r_{k,s}$: $n_{r,k,s} = 2n_{r-1,k,s} + k = k(2^r - 1) + s \cdot 2^r$.

Let $T(x, y, \mathbf{v}^k)$ be an arbitrary irredundant DNF of the support function $f(x, y, \mathbf{v}^k)$. Starting from the relation $\bar{f}(x, y, \mathbf{v}^k) = f(\bar{x}, \bar{y}, \mathbf{v}^k)$, define DNFs for the functions $f^r_{k,s}$, $r \geq 1$, $k \geq 2$, $s \geq 1$. Namely, denote by $T_{k,0}(x, y, \mathbf{v}^k)$ an irredundant DNF T of the function $f(x, y, \mathbf{v}^k)$, and by $T_{k,1}(x, y, \mathbf{v}^k)$ an irredundant DNF $T(\bar{x}, \bar{y}, \mathbf{v}^k)$ of the function $\bar{f}(x, y, \mathbf{v}^k)$. The DNFs

$$\bigvee_{\sigma_1+\cdots+\sigma_s=0(\bmod 2)} w_1^{\sigma_1}\cdots w_s^{\sigma_s} \quad \text{and} \quad \bigvee_{\sigma_1+\cdots+\sigma_s=1(\bmod 2)} w_1^{\sigma_1}\cdots w_s^{\sigma_s}$$

of the functions $f^0_{k,s}$ and $\bar{f}^0_{k,s}$ will be denoted by $T^0_{k,s,0}$ and $T^0_{k,s,1}$, respectively. If the DNFs $T^{r-1}_{k,s,0}$ and $T^{r-1}_{k,s,1}$ have already been constructed, then as the DNFs $T^r_{k,s,0}$ and $T^r_{k,s,1}$ we take, respectively, the DNFs $T_{k,0}(T^{r-1}_{k,s,1}(\mathbf{x}), T^{r-1}_{k,s,1}(\mathbf{y}), \mathbf{v}^k)$ and $T_{k,1}(T^{r-1}_{k,s}(\mathbf{x}), T^{r-1}_{k,s}(\mathbf{y}), \mathbf{v}^k)$. The DNFs $T^r_{k,s,0}$ and $T^r_{k,s,1}$ realize the functions $f^r_{k,s}$ and $\bar{f}^r_{k,s}$, and, by Lemma 1, are irredundant.

Lemma 2. *If $f(x, y, \mathbf{v}^k)$ is a support function and $k \leq s$, then* $\operatorname{Dim} f^r_{k,s} =$

$$= \operatorname{Dim} f^{r-1}_{k,s} + n_{k-2,k,s} = n_{0,k,s} + \cdots + n_{r-1,k,s}, \qquad Y(f^r_{k,s}) \geq 2^{\operatorname{Dim} f^r_{k,s} - 2^r \varepsilon(k)}.$$

The proof of the lemma is based on the fact that, by virtue of the linearity of the DNF $\mathfrak{A}$ and the nonlinearity of the DNF $\mathfrak{B}$ of the function $f(x, y, \mathbf{v}^k)$, the number

conjunctions $I(\mathfrak{B}^r_{k,s,0})$ in the d.n.f. $\mathfrak{B}^r_{k,s,0}$ grows, with the growth of $n_{r,k,s}$, much faster than $I(\mathfrak{A}^r_{k,s,0})$—the number of conjunctions in the d.n.f. $\mathfrak{A}^r_{k,s,0}$ of the function $f^r_{k,s}$.

Starting from Lemma 2, for any $\delta > 0$, any n, $n \geq n_s$, and any function $\mathcal{D}(n)$, $0 \leq \mathcal{D}(n) \leq n$, one can construct such a function $f^r_{k,s}$ that

$$n_{r,k,s}/n \geq 1 - \delta, \quad \operatorname{Dim} f^r_{k,s}/\mathcal{D}(n) \geq 1 - \delta, \quad Y(f^r_{k,s}) \geq 2^{\operatorname{Dim} f^r_{k,s} - \delta}.$$

Obviously, for the function

$$f_n(x^n) = f^r_{k,s}(x^{n_{r,k,s}})\, x_{n_{r,k,s}+1} x_{n_{r,k,s}+2} \cdots x_n$$

the relations

$$\operatorname{Dim} f_n / \mathcal{D}(n) \geq 1 - \delta, \quad Y(f_n) \geq 2^{\operatorname{Dim} f_n - \delta}$$

also hold. As was said in § 2, the proof of the theorem consists in constructing precisely such a function $f_n(x_1, \dots, x_n)$. To complete the proof fully, it remains to construct a support function.

5. Construction of the support function. "Neighboring" tuples

$$(\sigma_1, \dots, \sigma_{i-1}, 0, \sigma_{i+1}, \dots, \sigma_p)$$

and

$$(\sigma_2, \dots, \sigma_{i-1}, 1, \sigma_{i+1}, \dots, \sigma_p), \quad i = 1, \dots, p,$$

will be denoted by $\sigma^i(0)$ and $\sigma^i(1)$, and the unordered pair of these tuples (an "edge" of the p-dimensional unit cube) by σ^i. The collection of all pairs σ^i will be denoted by $\mathfrak{R}^p$. For a tuple $\sigma = (\sigma_1, \dots, \sigma_p)$ define

$$|\sigma| = \sigma_1 + \dots + \sigma_p \pmod 2,$$

and for a pair σ^i define

$$|\sigma^i| = \sigma_1 + \dots + \sigma_{i-1} + \sigma_{i+1} + \dots + \sigma_p \pmod 2.$$

We agree that $x^\alpha = x$ when $\alpha = 0$, and $x^\alpha = \bar{x}$ when $\alpha = 1$. We shall denote $x^{\alpha+\beta \pmod 2}$ by $x^{\alpha+\beta}$, $\alpha, \beta = 0, 1$.

Partition the p-dimensional unit cube, $p = 2^q - 1$, $q = 1, 2, \dots$, into

$$\frac{2^p}{p+1}$$

disjoint balls of radius 1. Let C^p be the set of centers of these balls, and let $\tau = (\tau_1, \dots, \tau_p)$ be an arbitrary tuple from C^p. Let

$$\pi(x, y, z, w) = [[unclear : displayedBooleanformuladefining\pi]].$$

Define the functions

$$R_{\sigma^i,\tau}(z^p, w^p) = \bigwedge_{\substack{j=1 \\ j \neq i}}^{p} z_j^{\sigma_j} w_j^{\sigma_j + \tau_j}, \quad \text{where } \sigma^i \in \mathfrak{R}^p,\ \tau \in C^p,\ p = 2^q - 1,\ q = 1, 2, \dots;$$

$$\varphi_\tau(z^p, w^p) = \bigvee_{\sigma^i \in \mathfrak{R}} R_{\sigma^i,\tau}(z^p, w^p);$$

$$\psi_\tau(x, y, z^p, w^p) = \bigvee R_{\sigma^i,\tau}(z^p, w^p)\,\pi(x, y, z_i^{|\sigma^i|+|\tau|}, w_i^{|\sigma^i|+\tau_i});$$

$$\xi(x, y, z^p, w^p) = \bigvee_{\tau \in C^p} \psi_\tau(x, y, z^p, w^p).$$

It turns out that the function $\xi(x, y, z^p, w^p)$, $p = 2^q - 1$, $q = 1, 2, ...$, will be a support function in x, y. Omitting consideration of the irredundant d.n.f.' s of this function, we note only that the relation

$$\bar{\xi}(x, y, z^p, w^p) = \xi(\bar{x}, \bar{y}, z^p, w^p)$$

follows from the following lemma on the functions $\varphi_\tau(z^p, w^p)$.

Lemma 3. $\varphi_\tau \cdot \varphi_\nu \equiv 0$ when $\tau \neq \nu$; $\tau, \nu \in C^p$, and

$$\bigvee_{\tau \in C^p} \varphi_\tau \equiv 1.$$

Let us note in passing that from this lemma there follows a simple inductive construction, for $q = 1, 2, ...$, of sets C^p, $p = 2^q - 1$, including such sets C^p as are not group codes.

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