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Abstract

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ASYMPTOTICS OF THE SOLUTION OF A MIXED PROBLEM FOR A FOURTH-ORDER EQUATION WITH A SMALL PARAMETER DEGENERATING INTO A SECOND-ORDER HYPERBOLIC EQUATION

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Let Q be a rectangle in two-dimensional space Q $(0 \leq t \leq T,\ 0 \leq x \leq l)$. Consider the following problem: in Q, determine a solution of the equation

$$\mathcal{L}_\varepsilon u \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^4 u}{\partial x^4} = f(t, x), \tag{1}$$

which satisfies the following initial and boundary conditions

$$u\Big|_{t=0} = 0, \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0, \tag{2}$$

$$u\Big|_{x=0} = 0, \qquad u\Big|_{x=l} = 0, \tag{3}$$

$$\frac{\partial u}{\partial x}\Big|_{x=0} = 0, \qquad \frac{\partial u}{\partial x}\Big|_{x=l} = 0,$$

where $\varepsilon > 0$ is a small parameter, and $f(t, x)$ is a given function. The conditions imposed on the function $f(t, x)$ will be formulated below.

The purpose of our note is to construct the asymptotics of the solution of the stated problem with respect to the small parameter. In constructing the asymptotics we shall use the method of M. I. Vishik and L. A. Lyusternik (^{1}).

Following (^{1}), for the solution we seek a representation of the form

$$u(t, x) = \left(w_0 + \sum_{i=1}^{n} \varepsilon^i w_i \right) + \left(v_0^1 + \sum_{i=1}^{n} \varepsilon^i v_i^1 \right) + \left(v_0^2 + \sum_{i=1}^{n} \varepsilon^i v_i^2 \right) +$$

$$+\left(\varphi_0^1+\sum_{i=1}^{n}\varepsilon^i\varphi_i^1\right)+\left(\varphi_0^2+\sum_{i=1}^{n}\varepsilon^i\varphi_i^2\right)+Z_\varepsilon^n(t,x), \tag{4}$$

where the functions w_i are determined by the first iterative process; v_i^1 and v_i^2 are functions of boundary-layer type and are determined by the second iterative process; the functions φ_i^1 and φ_i^2 are correction terms, and $Z_\varepsilon^n(t,x)$ is the remainder term.

First iterative process. We seek an approximate solution of equation (1) in the form

$$u(t,x)=\sum_{i=0}^{n}\varepsilon^i w_i. \tag{5}$$

Substituting the expression for $u(t,x)$ from (5) into equation (1) and collecting terms with like powers of ε, we obtain a recurrent system of equations.

$$\frac{\partial^2 w_0}{\partial t^2}-\frac{\partial^2 w_0}{\partial x^2}=f(t,x),$$

$$\frac{\partial^2 w_i}{\partial t^2}-\frac{\partial^2 w_i}{\partial x^2}=-\frac{\partial^4 w_{i-1}}{\partial x^4},\qquad i=1,2,\dots,n. \tag{6}$$

For each equation of system (6) a mixed problem is solved. For example, $w_0(t,x)$ is determined as the solution of the following problem:

$$\frac{\partial^2 w_0}{\partial t^2}-\frac{\partial^2 w_0}{\partial x^2}=f(t,x); \tag{7}$$

$$w_0\big|_{t=0}=0,\qquad \left.\frac{\partial w_0}{\partial t}\right|_{t=0}=0; \tag{8}$$

$$w_0\big|_{x=0}=0,\qquad w_0\big|_{x=l}=0. \tag{9}$$

Obviously, under the assumption of sufficient smoothness of the function $f(t,x)$, the solution of this problem has the form

$$w_0(t,x)=\int_0^t\int_0^l\left\{\frac{2}{l}\sum_{k=1}^{\infty}\frac{l}{k\pi}\sin\frac{k\pi}{l}(t-\tau)\sin\frac{k\pi}{l}x\sin\frac{k\pi}{l}\xi\right\}f(\tau,\xi)\,d\xi\,d\tau.$$

Let $f(t,x)$ be a function having $4n+2$ continuous derivatives with respect to x, and suppose that the function itself and its derivatives up to order $4n+1$, inclusive, vanish at the endpoints of the interval $[0,l]$. In addition, suppose that $f(t,x)$ has $2n-2$ continuous derivatives with respect to t.* Then, for the remaining equations of system (6), solving the analogous mixed problem, we determine the remaining functions $w_1, w_2, \dots, w_n$.

The second iterative process. We describe this process near the boundary $x=0$, since at $x=l$ an analogous picture is obtained.

To determine the function v_i^1 in the equation $\mathcal{L}_\varepsilon v = 0$, we make the change of variables $\dfrac{x}{\sqrt{\varepsilon}} = y,\ x = \sqrt{\varepsilon}\, y$. In the new variables the equation $\mathcal{L}_\varepsilon v = 0$ has the form

$$\mathcal{L}_\varepsilon v = \varepsilon^{-1}\left(\frac{\partial^4 v}{\partial y^4} - \frac{\partial^2 v}{\partial y^2} + \varepsilon\frac{\partial^2 v}{\partial t^2}\right) = 0. \tag{10}$$

We seek an approximate solution of equation (10) in the form

$$v(t,x) = \sum_{i=0}^{n} \varepsilon^i v_i^1. \tag{11}$$

Substituting the value $v(t,x)$ from (11) into (10) and collecting terms with equal powers of ε, we obtain the system of equations

$$\frac{\partial^4 v_0^1}{\partial y^4} - \frac{\partial^2 v_0^1}{\partial y^2} = 0,$$

$$\frac{\partial^4 v_i^1}{\partial y^4} - \frac{\partial^2 v_i^1}{\partial y^2} = -\frac{\partial^2 v_{i-1}^1}{\partial t^2}, \qquad i = 1, 2, \dots, n. \tag{12}$$

The first function of expansion (11) is determined from the following problem:

$$\frac{\partial^4 v_0^1}{\partial y^4} - \frac{\partial^2 v_0^1}{\partial y^2} = 0; \tag{13}$$

$$\frac{\partial}{\partial x}(w_0 + v_0^1)\Big|_{x=0} = 0. \tag{14}$$

The characteristic equation corresponding to equation (13) has two nonzero roots, -1 and 1. The fact that one of these roots is positive and the other negative ensures the regularity of the degenerate—

* The conditions imposed on $f(t, x)$ can be weakened.

Then the solution of equation (13) of boundary-layer type satisfying condition (14) has the form

$$\bar{v}_0^1 = \sqrt{\varepsilon} \left(\frac{\partial w_0}{\partial x} \bigg|_{x=0} \right) e^{-x/\sqrt{\varepsilon}}.$$

Obviously, the sum $w_0 + \bar{v}_0^1$ satisfies condition (14), but does not satisfy the condition

$$(w_0 + v_0^1)\big|_{x=0} = 0. \tag{15}$$

Therefore to the sum $w_0 + \bar{v}_0^1$ we add a correction term $\bar{\varphi}_0^1(t)$ so that the resulting sum $w_0 + \bar{v}_0^1 + \varphi_0^1$ satisfies conditions (14) and (15). Obviously,

$$\bar{\varphi}_0^1(t) = -\sqrt{\varepsilon} \left(\frac{\partial w_0}{\partial x} \bigg|_{x=0} \right).$$

In exactly the same way the functions $\bar{v}_1^1, \bar{v}_2^1, \dots, \bar{v}_n^1$ and $\bar{\varphi}_1^1, \bar{\varphi}_2^1, \dots, \bar{\varphi}_n^1$ are determined. We multiply the functions $\bar{v}_i^1$ and $\bar{\varphi}_i^1$ by a smoothing function. The resulting functions will be denoted by v_i^1 and φ_i^1 $(i = 0, 1, \dots, n)$.

To determine the functions v_i^2 $(i = 0, 1, \dots, n)$ in the equation $\mathcal{L}_\varepsilon v = 0$, one must make the change of variable $(l - x)/\sqrt{\varepsilon} = y$ and repeat all the arguments presented above. Thus, we have determined all the terms of expansion (4).

Theorem 1. If $u(t, x)$ is the solution of problem (1), (2), (3) and $f(t, x) \in L_2(Q)$, then

$$\|u\| \leq M\|f\|, \tag{16}$$

where M is a constant independent of ε, and the norm is understood in the metric of the space $L_2(Q)$.

We now estimate the remainder term $Z_\varepsilon^n(t, x)$. To do this, we apply the operator $\mathcal{L}_\varepsilon$ to both parts of expansion (4). We obtain

$$\mathcal{L}_\varepsilon Z_\varepsilon^n = \varepsilon^{n+1/2} \psi(t, x, \varepsilon),$$

where ψ is a bounded function. Therefore

$$\mathcal{L}_\varepsilon Z_\varepsilon^n = O(\varepsilon^{n+1/2}).$$

If we take estimate (16) into account, we obtain

$$Z_\varepsilon^n = O(\varepsilon^{n+1/2}).$$

Thus, the following has been proved:

Theorem 2. If $f(t,x)$ has $4n+2$ continuous derivatives with respect to x, the function itself and its derivatives with respect to x up to order $(4n+1)$ inclusive vanish at the ends of the interval $[0,l]$, and, in addition, $f(t,x)$ has $2n-2$ continuous derivatives with respect to t, then the solution of problem (1), (2), (3) can be represented in the form (4), and $Z_\varepsilon^n(t,x)$ tends to zero in the sense of the metric of $L_2(Q)$ as $\varepsilon \to 0$ like $\varepsilon^{n+1/2}$.

In an analogous way it is shown that if the degenerate equation is parabolic, i.e., if the equation has the form

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial^4 u}{\partial x^4} = f(t,x),$$

then its solution under the conditions

$$u\big|_{t=0} = 0, \qquad u\big|_{x=0} = 0, \qquad u\big|_{x=l} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0$$

is representable in the form (4).

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REFERENCES

1. M. I. Vishik, L. A. Lyusternik, *UMN*, **12**, no. 5 (77) (1957).

Note: Figure translations are in progress. See original paper for figures.

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