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Abstract

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MATHEMATICS

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ON SOME APPLICATIONS OF PROJECTION SPECTRA TO THE THEORY OF TOPOLOGICAL SPACES

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In this paper a summary is given of certain results proved by direct application of the methods given in my papers ([1–3]); these methods essentially belong to the theory of projection spectra. Some of these results are contained in the papers of Nagami ([6,7]), published later and using essentially the same methods.

§ 1. In paper ([2]) the following order was considered in the set of all closed coverings, going back already to ([4]) and even to ([5]): let α and α' be two closed coverings; we say that α' **follows** α if α' is inscribed in α and each element of the covering α is the sum of all elements of the covering α' contained in it. A covering α is called a **partition** or a **canonical covering** if it is locally finite and its elements are the closures of pairwise disjoint open sets of the space X. In paper ([2]), and partly already in ([1]), the following notions were introduced.

A set $\mathfrak{A} = \{\alpha\}$ of (arbitrary) coverings of any completely regular space is called: 1) **refining**, if for every point $x \in X$ and every neighborhood Ox there is an α_0 such that the star of the point x in the covering α_0 is contained in Ox; 2) **strongly refining**, if in the preceding definition one replaces the point x and its neighborhood Ox by an arbitrary closed $F \subseteq X$ and its neighborhood OF; 3) **cofinally refining**, if for every open covering ω of the space X there is an $\alpha \in \mathfrak{A}$ inscribed in it.

In ([1]) and further in ([3]) the following proposition was proved: under the order established in the set of closed coverings, the set of all finite canonical and, a fortiori, the set of all canonical coverings of a completely regular (respectively normal) space X will be a directed refining (respectively strongly refining) set.

To each directed set of canonical coverings $\mathfrak{A} = \{\alpha\}$ of the space X, with the natural mappings $\omega_\alpha^{\alpha'} : \alpha' \to \alpha$ for $\alpha' > \alpha$, there corresponds the spectrum $S_{\mathfrak{A}} = \{|\alpha|, \omega_\alpha^{\alpha'}\}$ of the nerves $|\alpha|$ of these coverings. The spectrum $\dot{S}_{\mathfrak{A}} = \{\dot{\alpha}, \dot{\omega}_\alpha^{\alpha'}\}$, which we shall call the **complete partition** of the spectrum $S_{\mathfrak{A}}$, consists of the complexes $\dot{\alpha}$, where $\dot{\alpha}$ is the zero-dimensional complex composed of all vertices of the complex $|\alpha|$, with the same projections as in $S_{\mathfrak{A}}$.

Suppose that in the space X a refining directed set of partitions $\mathfrak{A} = \{\alpha\}$ is given. Then, by the method of paper ([1]), a space $\dot{X}_0$ is constructed which is an everywhere dense subset of the limit space $\dot{X} = \dot{S}_{\mathfrak{A}}$, and a perfect irreducible mapping f of the space $\dot{X}_0$ onto the whole space X. Let us recall how the space $\dot{X}_0$ was constructed. A thread $\dot{x} = \{e_\alpha\} \in \dot{X}$ was called marked if the sets $A^\alpha \in \alpha$ corresponding to these vertices have a nonempty intersection. As a consequence of refinement

sequence $\mathfrak{A}$, this intersection consists of a single point $x \in X$. The totality of all marked threads $\dot{x} = \{e_\alpha\}$ of the spectrum $\widetilde{S}_{\mathfrak{A}}$ forms, by definition, the space $\dot{X}_0$, and $\operatorname{ind} \dot{X}_0 = 0$.

Let $\dot{x} = \{e_\alpha\} \in \dot{X}_0$ be arbitrary; then the corresponding $A^\alpha \in \alpha$ have an intersection consisting of a single point $x \in X$. We obtain a mapping $f\dot{x} = x : \dot{X}_0 \to X$. As in (1), it is verified that this mapping is onto all of X, and that f is a perfect* irreducible mapping of the space $\dot{X}_0$ onto the (completely regular) X.

It is easy to prove:

Lemma. If a T_1-space Y is the image of a T_1-space X under a closed continuous mapping f, then the space Y is normal if and only if the space X is normal on every closed set** A of the form $A = f^{-1}fA$.

Now it is proved (see (1)):

Theorem 1. Each of the following conditions is necessary and sufficient for the normality of the T_1-space X:

A. The set of all (respectively, of all finite) partitions of the space X is directed and strongly refining.

B. The space X (of weight τ) is the image under a perfect (irreducible) mapping $f : X_0 \to X$ of a completely regular and zero-dimensional, in the sense of $\operatorname{ind} X_0 = 0$, space $X_0 \subseteq D^\tau$, which, for every set $A \subseteq X$ of the form $A = f^{-1}fA$, satisfies the condition $\operatorname{Ind}_A X = 0$.

The second part of this theorem, which strengthens the main result of the paper (1), is naturally supplemented by the proposition proved in (3).

Theorem 2. Each of the following conditions is necessary and sufficient in order that the T_1-space X be paracompact:

a) The set of all partitions of the space X is directed and confinally refining.

b) The space X is the image of a perfectly zero-dimensional space*** X_0 under a perfect (irreducible) mapping****.

If the set of partitions $\mathfrak{A}$ is confinally refining, then X is a paracompact space homeomorphic to the limiting space $\widetilde{S}_{\mathfrak{A}}$ (see (3)); the perfectly zero-dimensional space $\dot{X}_0$, of which the space $\widetilde{S}$ is an irreducible perfect image, in this case coincides with the whole space $\dot{X} = \dot{\widetilde{S}}_{\mathfrak{A}}$.

Definition 1. A spectrum $S_2 = \{|\beta|, \omega_\beta^{\beta'}\}$ is called an **amplification** of the spectrum $S_1 = \{|\alpha|, \omega_\alpha^{\alpha'}\}$ if the following conditions are satisfied: 1) there exists a one-to-one similar mapping $\alpha \to \beta_\alpha$ of the directed set of indices $A = \{\alpha\}$ onto the directed set of indices $B = \{\beta\}$ (here A may be regarded as a multiplication of B); 2) the set of all vertices $\dot{\beta}_\alpha$ of the complex $|\beta_\alpha|$ coincides with the set of all vertices $\dot{\alpha}$ of the complex $|\alpha|$, and $|\beta_\alpha| \subseteq |\alpha|$.

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* A mapping $f : X \to Y$ is called **perfect** if it is continuous, closed, and bicompact in the sense that the inverse images $f^{-1}y$ of all points $y \in Y$ are bicompact.

** We call a space X **normal on a closed set** $A \subseteq X$ if, for every neighborhood OA of this set, there is a neighborhood $O'A$ such that $[O'A] \subseteq OA$.

*** A T_1-space is called **perfectly zero-dimensional** if it is regular and, into each of its open covers, one can inscribe a cover consisting of pairwise nonintersecting open sets.

**** Every regular space X of weight τ is the image of a completely regular space $X_0 \subseteq D^\tau$ under a continuous mapping f. I do not know whether this mapping can be assumed perfect, or at least closed. The existence of the mapping f follows (with the aid of all the same methods) from the fact that in every regular space (and only in a regular space) the set of all finite partitions is directed and refining.

Definition 2. Two spectra S_1 and S_2 are called **equivalent** (or belonging to one and the same class) if their full refinements $\dot{S}_1$ and $\dot{S}_2$ are isomorphic.

Theorem 3. If the spectrum S_2 is an enlargement of the spectrum S_1, then there exists a single-valued irreducible perfect mapping*
$\pi : \widetilde{\dot{S}}_1 \to \widetilde{\dot{S}}_2^*$. In particular, there exists an irreducible perfect ("standard") mapping π_X of the limiting space $\widetilde{\dot{S}} = \dot{X}$ (where $\dot{S}$ is the full refinement of the spectrum S) onto the space $\widetilde{S} = X$ of the spectrum $S = \{\alpha, \omega_\alpha^a\}$. If two spectra S_1 and S_2 are equivalent, then and only then their spaces X_1 and X_2 are mapped onto each other multivalently, perfectly and irreducibly (in the sense of paper ([3])) by the formula

$$f = \pi_{X_2} f^{\cdot} (\pi_{X_1})^{-1},$$

where $f^{\cdot}$ is some homeomorphism of the space $\widetilde{\dot{S}}_1 \equiv \widetilde{\dot{S}}_2 \equiv \dot{X}$ onto itself, and π_{X_2}, π_{X_1} are the corresponding standard mappings of $\dot{X}$ onto X_2 and X_1; these mappings may be interpreted as a passage from the spectrum $\dot{S}_i$ to the spectrum S_i, $i = 1, 2$, which is an enlargement of the spectrum $\dot{S}_i$.

§ 2. **Definition 1.** Let in X (an arbitrary normal space) there be a directed set $\mathfrak{A}$ of decompositions of multiplicity $\leq n+1$: a) refining; b) strongly refining; c) cofinally refining (in the last case the space X is paracompact). Then we say that the approximation dimension dX (in case a)), the large approximation dimension DX (in case b)), the cofinal approximation dimension ΔX (in case c)) does not exceed n. Naturally, dX itself, respectively DX and ΔX, is the least $n \geq 0$ (if it exists) satisfying this condition. If there is no such n, then the corresponding dimension is taken to be equal to ∞.

Entirely by the methods of paper ([2]) one proves

Theorem 4. In any normal space the following relations between dimensions hold:

$$\operatorname{ind} X \leq dX, \qquad \operatorname{ind} X \leq \operatorname{Ind} X \leq DX, \qquad \dim X \leq \Delta X.$$

If the dimension $\dim X = \Delta X$, then

$$\operatorname{ind} X \leq \operatorname{Ind} X \leq \dim X = \Delta X.$$

Definition 2 (basic) ([2]). A space X is called **perfectly n-dimensional** if $\dim X = \Delta X = n$.

Corollary of Theorem 4. If a strongly paracompact space X is perfectly n-dimensional, then

$$\operatorname{ind} X = \operatorname{Ind} X = \dim X = \Delta X = n.$$

Theorems 2 and 3 of paper ([2]) carry over to arbitrary normal spaces with the aid of the constructions given above (taken from papers ([1,2])). We obtain the propositions:

Theorem 5′. A completely regular space X (of weight τ) has $dX \leq n$ if and only if there exists $X_0 \subseteq D^\tau$ (and hence $\operatorname{ind} X_0 = 0$) and an $(n+1)$-fold irreducible perfect mapping f of the space X_0 onto X.

Theorem 5″. $\Delta X \leq n$ if and only if X is a perfect $(n+1)$-fold image of some perfectly zero-dimensional space** X_0.

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* In this case the mapping π may be interpreted as a passage from the spectrum S_1 to the spectrum S_2 in the sense of paper ([3]).

** See papers ([1,3]).

We note that from the results of Morita [6] and Theorem 6 of the present paper there follows

Theorem 6. *A metric space X: $\dim X = n$ is perfectly n-dimensional.*

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Note: Figure translations are in progress. See original paper for figures.