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Abstract

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MATHEMATICS

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ON A GENERALIZATION OF S. N. BERNSTEIN'S THEOREM ON THE FORM OF NONNEGATIVE TRIGONOMETRIC POLYNOMIALS TO THE CASE OF AN ARBITRARY NUMBER OF RELATIONS BETWEEN THE COEFFICIENTS OF THE POLYNOMIALS

(Presented by Academician V. I. Smirnov on 1 VI 1962)

1. S. N. Bernstein ([1]) is responsible for a remarkable theorem, according to which, if the nonnegative trigonometric sum

$$s_n(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + \cdots + a_n \cos n\theta + b_n \sin n\theta \geq 0 \tag{1}$$

satisfies the relations

$$\int_0^{2\pi} s_n(\theta) F_1(\theta)\, d\theta = \omega_1; \qquad \int_0^{2\pi} s_n(\theta) F_2(\theta)\, d\theta = \omega_2, \tag{2}$$

then the absolute minimum of the integral

$$\int_0^{2\pi} s_n(\theta)\varphi(\theta)\, d\theta \tag{3}$$

can always be attained by a trigonometric sum having only real roots (of even multiplicity). Here $F_1(\theta)$ and $F_2(\theta)$ are arbitrary prescribed integrable functions, ω_1 and ω_2 are prescribed constants, and $\varphi(\theta) \geq 0$ is likewise a prescribed integrable function. Thus, among the sums realizing the minimum, there always exists a sum of the form

$$s_n(\theta) = P^2(\theta),$$

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where all roots of the trigonometric polynomial $P(\theta)$ are real.

On the basis of this theorem, S. N. Bernstein considered a number of extremal problems in the theory of nonnegative trigonometric polynomials. Subsequently the theorem was successfully applied by A. G. Nyrkova ([2,3]).

2. Let us consider the analogous problem of finding a function (1) realizing the minimum of the integral (3), but we shall not restrict the number of relations imposed on the coefficients; i.e., let $s_n(\theta)$ satisfy s relations

$$\int_0^{2\pi} s_n(\theta) F_j(\theta)\, d\theta = a_0 A_{0j} + \sum_{k=1}^{n} (a_{kA_{kj}} + b_{kB_{kj}}) = \omega_j, \qquad j = 1, \dots, s, \tag{4}$$

with the previous notation. We additionally assume that the linear forms

$$a_0 A_{0j} + \sum_{k=1}^{n} (a_{kA_{kj}} + b_{kB_{kj}}), \quad j = 1, \dots, s,$$

and

$$a_0 + \sum_{k=1}^{n} (a_k \cos k\theta_i + b_k \sin k\theta_i)$$

in the coefficients a_k and b_k are linearly independent (θ_i are all real, pairwise unequal roots of the equation $s_n(\theta) = 0$, belonging to the interval $[0, 2\pi)$). Under this assumption, among the relations (4) there must not occur relations of the form

$$s_n(\eta) = s_n'(\eta) = \dots = s_n^{(2p-1)}(\eta) = 0, \tag{5}$$

where η is a prescribed point; but this does not diminish the generality of the problem, since, if among the conditions (4) there are 2ρ conditions (5), then the problem reduces to the differen-

to the search for a nonnegative sum $s_{n-p}(\theta)$ of order not higher than $n - p$, satisfying $s - 2p$ relations (the conditions (5) being excluded) and realizing the minimum of the integral

$$\int_0^{2\pi} s_{n-p}(\theta)\, \varphi(\theta)\, \sin^{2p} \frac{\theta - \eta}{2}\, d\theta.$$

3. The sums (1), evidently, can always be represented in the form

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$$s_n(\theta) = P^2(\theta)\, q(\theta),$$

where the polynomial $P(\theta)$ has only real roots, while the polynomial $q(\theta) \geq \rho^2 > 0$ for all real θ.

Let r be the order of the polynomial $q(\theta)$. Then, applying arguments analogous to those which S. N. Bernstein used in article (1), it is easy to verify that, for the extremal function $s_n(\theta)$, one will have

$$r < s/2.$$

We shall establish additional relations (besides the given s relations (4)) which must be satisfied by the coefficients of the extremal function $s_n(\theta)$. Since all the roots of $P(\theta)$ are real, we have

$$P(\theta) = A \sin\frac{\theta - \theta_1}{2} \cdots \sin\frac{\theta - \theta_m}{2},$$

where A is a certain constant.

The number of desired relations between the coefficients of the function $s_n(\theta)$, consequently, is equal to $m + 2r + 1 - s$. Denote $\tau = m + 2r - s$, and let $\tau \geq 0$.

4. For definiteness, suppose that m is an even number ($m = 2\mu$). Then

$$P(\theta) = x_0 + x_1 \cos\theta + y_1 \sin\theta + \cdots + x_\mu \cos\mu\theta + y_\mu \sin\mu\theta.$$

Let $\psi(\theta)$ denote any of the trigonometric polynomials of order not higher than $m + r$, satisfying the conditions

$$\int_0^{2\pi} \psi(\theta)\, F_j(\theta)\, d\theta = 0, \qquad j = 1, 2, \dots, s,$$

$$\psi(\theta_i) = B_i^2 > 0, \qquad i = 1, 2, \dots, m,$$

where B_i^2 are arbitrarily chosen real numbers, and θ_i are the roots of the polynomial $P(\theta)$. Taking into account the restrictions imposed on the conditions (4), we may assert that such polynomials $\theta(\psi)$ always exist.

Let s also be an even number ($s = 2\nu$). Consider the polynomials

$$t_1^{(l)}(\theta) = \sin l\theta\, (a_{0l} + a_{1l} \cos\theta + \cdots + b_{\nu l} \sin\nu\theta),$$

whose coefficients a_{kl} and b_{kl}, for each fixed value of l (l, $1 \leq l \leq \tau/2$, an integer), satisfy the equations

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$$\int_0^{2\pi} P(\theta)\, t_1^{(l)}(\theta)\, F_j(\theta)\, d\theta = 0, \qquad j = 1, 2, \dots, s,$$

i.e.

$$a_{0l} e_{0j}^{(l)} + \sum_{k=1}^{\nu} \left(a_{kl} e_{kj}^{(l)} + b_{kl} f_{kj}^{(l)} \right) = 0, \qquad j = 1, \dots, s, \tag{6}$$

where

$$e_{kj}^{(l)} = \int_0^{2\pi} P(\theta) F_j(\theta) \sin l\theta \cos k\theta \, d\theta;$$

$$f_{kj}^{(l)} = \int_0^{2\pi} P(\theta) F_j(\theta) \sin l\theta \sin k\theta \, d\theta.$$

We now form the sum

$$s_n(\theta) = P(\theta)[P(\theta) q(\theta) - \lambda t_1^{(l)}(\theta)] + \lambda \psi(\theta).$$

We have

$$\int_0^{2\pi} s_n(\theta) \varphi(\theta)\, d\theta = \int_0^{2\pi} s_n(\theta)\varphi(\theta)\, d\theta - \lambda \left[a_{0l} g_0^{(l)} + \sum_{k=1}^{\nu} \left(a_{kl} g_k^{(l)} + b_{kl} h_k^{(l)} \right) - \int_0^{2\pi} \psi(\theta)\varphi(\theta)\, d\theta \right], \tag{7}$$

where

$$g_k^{(l)} = \int_0^{2\pi} P(\theta) \varphi(\theta) \sin l\theta \cos k\theta \, d\theta;$$

$$h_k^{(l)} = \int_0^{2\pi} P(\theta) \varphi(\theta) \sin l\theta \sin k\theta \, d\theta.$$

If the determinant

$$\Delta^{(l)} = \begin{vmatrix} e_{01}^{(l)} & \dots & e_{\nu 1}^{(l)}, & f_{11}^{(l)} & \dots & f_{\nu 1}^{(l)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e_{0s}^{(l)} & \dots & e_{\nu s}^{(l)}, & f_{1s}^{(l)} & \dots & f_{\nu s}^{(l)} \\ g_0^{(l)} & \dots & g_\nu^{(l)}, & h_1^{(l)} & \dots & h_\nu^{(l)} \end{vmatrix} \neq 0,$$

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then we can always require that the equality

$$a_{0l}g_0^{(l)} + \sum_{k=1}^{\nu} \left(a_{kl}g_k^{(l)} + b_{kl}h_k^{(l)}\right) = \omega^2$$

be satisfied, whatever $\omega^2 > 0$ may be. Choosing ω^2 sufficiently large so that the expression in square brackets in equality (7) is positive, we obtain, for $\lambda > 0$,

$$\int_0^{2\pi} \tilde{s}(\theta)\varphi(\theta)\,d\theta < \int_0^{2\pi} s_n(\theta)\varphi(\theta)\,d\theta.$$

Further, taking into account that

$$\tilde{s}_n(\theta_i) = \lambda B_i^2 > 0,$$

it is easy to show that we can always choose $\lambda > 0$ sufficiently small so that the sum $\tilde{s}_n(\theta)$ is nonnegative for all θ.

Consequently, in order that the polynomial $s_n(\theta)$ be extremal, all determinants $\Delta^{(l)}$ must be equal to zero, but then

$$\int_0^{2\pi} P(\theta)t_1^{(l)}(\theta)\varphi(\theta)\,d\theta = 0, \qquad l = 1, 2, \dots, \frac{\tau}{2}. \tag{8}$$

Similarly we obtain the system of equations

$$\int_0^{2\pi} P(\theta)t_2^{(l)}(\theta)\varphi(\theta)\,d\theta = 0, \qquad l = 0, 1, \dots, \frac{\tau}{2}, \tag{9}$$

where the polynomials

$$t_2^{(l)}(\theta) = \cos l\theta\,(a_{0l} + a_{1l}\cos\theta + \dots + b_{\nu l}\sin\nu\theta)$$

for each value of l satisfy the relations

$$\int_0^{2\pi} P(\theta)t_2^{(l)}(\theta)F_j(\theta)\,d\theta = 0, \qquad j = 1, 2, \dots, s.$$

The required relations (8) and (9) for the coefficients of extremal polynomials $s_n(\theta)$ have thus been established.

In the case of odd s $(s = 2\nu + 1)$, the sums $t_1^{(l)}(\theta)$ and $t_2^{(l)}(\theta)$ are equal to

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$$t_1^{(l)}(\theta) = \sin\frac{2l+1}{2}\theta\left(a_{0l}\cos\frac{\theta}{2} + b_{0l}\sin\frac{\theta}{2} + ... + b_{\nu l}\sin\frac{2\nu+1}{2}\theta\right),$$

$$t_2^{(l)}(\theta) = \cos\frac{2l+1}{2}\theta\left(a_{0l}\cos\frac{\theta}{2} + b_{0l}\sin\frac{\theta}{2} + ... + b_{\nu l}\sin\frac{2\nu+1}{2}\theta\right),$$

$$0 \le l \le \frac{\tau-1}{2}.$$

All the remaining arguments remain valid.

5. As an application of the results obtained, we give the solution of the following problem:

Problem. Determine the minimum

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} s_n(\theta)\,d\theta,$$

if the values

$$s_n(0) = A^2, \qquad s_n'(0) = B, \qquad s_n''(0) = C$$

are given ($A^2 \ge 0$, B and C are given numbers). For definiteness we assume that n is an even number.

Then for $A^2 > 0$ the minimum a_0 will be computed either by the formula

$$\min a_0 = \frac{3}{(n-1)(n+1)(n+3)}\left[\frac{3n^2+6n-4}{4}A^2 + \frac{3\cdot 5}{n(n+2)A^2}\left(C - \frac{B^2}{2A^2}\right)^2 - \frac{3(n^2+2n+2)B^2}{2n(n+2)A^2} + 5C\right], \tag{10}$$

if the condition

$$C \le \frac{B^2}{2A^2} - \frac{n^2+2n+2}{2\cdot 5}A^2,$$

is satisfied, or by the formula

$$\min a_0 = \frac{2}{5n(n+1)(n+2)}\left[(3n^2+6n+1)A^2 + 2\cdot 5C\right], \tag{11}$$

if

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$$C > \frac{B^2}{2A^2} - \frac{n^2+2n+2}{2\cdot 5}A^2.$$

From formulas (10) and (11) one obtains the result of S. N. Bernstein ([1]) (when $s_n(0)$ and $s'_n(0)$ are given), if one sets

$$C = \frac{B^2}{2A^2} - \frac{n(n+2)}{2\cdot 3}A^2,$$

and the result of A. G. Nyrkova ([2]) (when $s_n(0)$ and $s''_n(0)$ are given), if one sets

$$B = 0.$$

If, however, $A^2 = 0$ (then also $B = 0$), then

$$\min a_0 = \frac{2\cdot 3}{n(n+1)(n+2)}\,C.$$

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CITED LITERATURE

[1] S. N. Bernstein, *Collected Works*, 1, Publishing House of the Academy of Sciences of the USSR, 1952, p. 472.
[2] A. G. Nyrkova, *Proceedings of the Leningrad Industrial Institute*, Section of Physics and Mathematics, issue 1, 5 (1939).
[3] A. G. Nyrkova, ibid., No. 3, 50 (1941).

Note: Figure translations are in progress. See original paper for figures.

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