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Abstract

Full Text

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ON APPROXIMATION ON CLOSED DOMAINS AND ON ZERO SETS*

(Presented by Academician A. N. Kolmogorov on 27 XI 1961)

S. N. Mergelyan established ([1]) that if G is a finitely connected domain, moreover such that: 1) the complement $C\bar{G}$ of the closure $\bar{G}$ of this domain consists of a finite number of domains; 2) G coincides with the set of interior points of its closure $\bar{G}$, then the following holds:

Theorem A. *Every function analytic inside G and continuous on $\bar{G}$ can be uniformly approximated on $\bar{G}$, with any prescribed accuracy, by rational fractions.*

Let us note that if the domains g_n, complementary to $\bar{G}$, are at a positive distance from one another, then this assertion follows from an earlier result of M. V. Keldysh ([2]).

Will this theorem remain true if no restrictions are imposed on the domain G (apart from the natural condition that G coincide with the set of interior points of its closure), or if even the condition of finite connectedness of $\bar{G}$ is dropped, retaining this condition only for G? Below this question is answered in the negative. For a certain class $B^{(1/3)}$ of infinitely connected domains G, a complete structural characterization is given of the functions $f(z)$ $(z \in \bar{G})$ that admit uniform approximation on $\bar{G}$ by rational fractions.

1. An example of a simply connected domain G for which Theorem A is not true. Take a function $\varphi(z)$, distinct from a constant, defined and continuous in the whole extended complex plane Z, analytic outside some bounded, nowhere dense continuum E in Z with connected complement, and such that $\max|\varphi(z)| = 1$. As $\varphi(z)$ one may take, for example, the function

$$l \iint_E \frac{d\xi\, d\eta}{\xi + i\eta - z}$$

(see ([3]), p. 330), where E is a bounded nowhere dense continuum in Z with connected complement, having positive planar Lebesgue measure, and l is a constant.

Let the point $a \in E$ be such that $|\varphi(a)| = 1$; let the circle σ_0 be chosen so that $E \subset \bar{\sigma}_0$ and on the boundary c_0 of this circle there is a single point $z_0 \in E$, with $z_0 \neq a$. Take a system of open circles $\{\sigma_n\}_{n=1}^{\infty}$ with the following properties: 1) $\sigma_n \subset \sigma_0 \setminus E$ $(n = 1, 2, ...)$; 2) on the boundary c_n of the circle σ_n there lies one and only one point $z_n \in E$, with $z_n \neq a$; 3) any two circles σ_n, σ_m $(m \neq n)$ are

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at a positive distance from each other and from c_0; 4) in every neighborhood of each point $z \in E$ there is an infinite number of circles σ_n; 5) if r_n $(n \geq 0)$ denotes the distance from a to c_n, and $|c_n|$ the length of the curve c_n, then

$$\sum_{n=1}^{\infty} |c_n|/r_n < 1/4.$$

Obviously, the set

$$G = \sigma_0 \setminus \left(\bigcup_{n \geq 1} \bar{\sigma}_n \right)$$

is a simply connected domain. The set E forms part of the boundary of this domain, so that $\varphi(z)$ is analytic in G and continuous on $\bar{G}$. Choose a natural–

* The result of the paper was reported at the IV All-Union Mathematical Congress in Leningrad in July 1961.

so large that for $z \in c = \{z: \ z \in c_0, \ |z - z_0| > \frac{1}{3} r_0\}$ one has $|\varphi(z)|^p < r_0/10|c_0|$ (this can be done, since $|\varphi(z)| < 1$ if $z \in c_0$, $z \neq a$). We shall show that for $f(z) = [\varphi(z)]^p$ and any rational function $R(z)$,

$$\max_{\overline{G}} |f(z) - R(z)| > r_0/|c_0|. \tag{1}$$

Assuming the contrary, for some rational function $R(z)$ we have: a) for $z \in \overline{G}$, $|R(z)| \leq 1 + r_0/|c_0| < 4/3$; b) for $z \in c$,

$$|R(z)| < r_0/10|c_0| + r_0/|c_0| = 11 r_0/10|c_0|;$$

c) $|R(a)| \geq 1 - r_0/|c_0| > 5/6$. If $\sigma_{n_1}, \sigma_{n_2}, \dots, \sigma_{n_k}$ are those of the circles σ_n $(n \geq 1)$ inside which poles of the fraction $R(z)$ have fallen, then

$$|R(z)| = \left| \frac{1}{2\pi i} \left[\int_{c_0} - \sum_{j=1}^{k} \int_{c_{n_j}} \right] \frac{R(z)\, dz}{a - z} \right| \leq \frac{1}{2\pi} \left[\int_{c} + \int_{c_0 \setminus c} + \sum_{n=1}^{\infty} \int_{c_n} \right] \frac{|R(z)|\, ds}{|a - z|}$$

$$\leq \frac{1}{2\pi} \left[\frac{11 r_0}{10|c_0|} \frac{|c_0|}{r_0} + \frac{4}{3} \frac{1}{r_0} \pi \frac{r_0}{30} + \sum_{n=1}^{\infty} \frac{4}{3} \frac{|c_n|}{r_n} \right] < \frac{11}{20\pi} + \frac{1}{45} + \frac{2}{3\pi} \frac{1}{4} < \frac{1}{3},$$

which contradicts inequality c): $|R(a)| > 5/6$. Thus, (1) is proved. Hence the following is true:

Theorem 1. *There exists a simply connected domain G (coinciding with the set of interior points of its closure $\overline{G}$) for which theorem A does not hold.*

2. Definitions and auxiliary propositions. Let G be a domain of the extended plane $\overline{Z}$, coinciding with the set of interior points of its closure $\overline{G}$; let

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Γ be the boundary of the domain G; $\Gamma = \overline{G} \setminus G$; let $\{g_n\}$ be the collection of domains adjacent to $\overline{G}$ ($\bigcup_n g_n = \overline{Z} \setminus \overline{G}$); let γ_n be the boundary of the domain g_n; $\tilde{\Gamma} = \bigcup_n \gamma_n$; and let K_p be a connected component of the set $\tilde{\Gamma}$ ($\bigcup_p K_p = \tilde{\Gamma},\ K_p \cap K_q$ empty for $p \neq q;\ p, q = 0, 1, \ldots$).

Let us introduce the set $\Lambda(G)$. It consists of those and only those points of the plane $\overline{Z}$ in every neighborhood of each of which there are points of infinitely many domains g_n. The set $\Lambda(G)$ will play an important role in what follows. Obviously, the closed domain $\overline{G}$ is finitely connected if and only if $\Lambda(G)$ is empty. Let us note here that $\Lambda(G)$ may have no points in common with $\tilde{\Gamma}$. An example of such a domain is obtained from the domain G constructed in § 1 if we concentrically enlarge the circle σ_0 and reduce the circles σ_n with $n \geq 1$.

It is easy to prove

Theorem 2. *In order that a set E coincide with the set $\Lambda(G)$ for some domain $G \subset \overline{Z}$, it is necessary and sufficient that: 1) E be closed; 2) E be entirely contained in the boundary of one of the domains adjacent to this set.*

Definition 1. A bounded domain G belongs to the class $B(\alpha)$ ($\alpha > 0$) if the following conditions are satisfied: 1) for every p the continuum K_p is the sum of a finite number of sets γ_n and is at a positive distance from the remaining part $\Gamma \setminus K_p$ of the boundary of the domain G; 2) if d_p denotes the diameter of the continuum K_p ($d_p > 0$!), then $\sum_p d_p^\alpha < \infty$; 3) there exist positive numbers $P = P(G)$ and $c = c(G)$ such that, for $p \geq P$, the cd_p^α-neighborhood of the continuum K_p contains no points of the set $\Gamma \setminus K_p$.

Obviously, any domain G with finitely connected closure $\overline{G}$ belongs to all classes $B(\alpha)$ ($\alpha > 0$); if $\alpha < \beta$, then $B(\alpha) \subset B(\beta)$. Let us also note that theorem 2 remains valid if one requires of the domain G that it belong to all classes $B(\alpha)$ with $\alpha > 0$ (in this case, of course, one must require of E ···

boundedness). This is proved by a construction analogous to that carried out in §1. As usual, we shall call a bounded domain G a domain with finite perimeter if γ_n are rectifiable curves and

$$\sum_n \operatorname{mes}_1 \gamma_n < \infty.$$

Definition 2. A functional property X is called regular if the following conditions are satisfied for it: 1) every function possessing property X on a set $E \subset Z$ is continuous on E; 2) if $f(z)$ possesses property X on E, then it possesses this property also on every set $F \subset E$; 3) if $f_1(z)$ and $f_2(z)$ possess property X on E, then their product $f_1(z) \cdot f_2(z)$, and also the linear combination $\alpha f_1(z) + \beta f_2(z)$ with real nonnegative numbers α, β, possess the same property X on E; 4) if $f(z)$ is continuous in the extended complex plane $\overline{Z}$, possesses property X on a bounded closed set $E \subset Z$, and is analytic outside the set E, then $f(z)$ possesses

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property X in the finite plane Z; 5) every function $f(z)$ uniformly differentiable* on a bounded set E possesses property X on E.

Examples of regular properties are: 1) continuity, 2) uniform differentiability, 3) belonging to the class $\mathrm{Lip}\,\alpha$. The last follows from the following lemma (which generalizes the theorem on p. 689 of [4]):

Lemma 1. *If $f(z)$ is analytic in an open set G, not containing the point ∞ on its boundary Γ, is continuous on $\overline{G}$ and on $\Gamma = \overline{G} \setminus G$ satisfies the condition* $\mathrm{Lip}_M\,\alpha$ $(0 < \alpha \leqslant 1)$, *then also on $\overline{G}$, $f \in \mathrm{Lip}_M\,\alpha$ with the same constant M.*

For what follows we shall also need

Lemma 2. *If $f(z)$ is uniformly differentiable on a bounded set E, analytic outside E** and everywhere continuous in $\overline{Z}$, then $f(z) \equiv \mathrm{const}$.*

Definition 3. The X-capacity of a bounded set $E \subset Z$ is the quantity

$$\gamma_X(E) = \sup_{f \in M(X,E)} \left\{ |\lim_{z\to\infty} z f(z)| \right\} = \sup_{f \in M(X,E)} \left\{ \left| \int_L f'(z)\,dz \right| \right\}.$$

Here L is a rectifiable Jordan contour enclosing E, and $M(X,E)$ is the class of functions $f(z)$ with the properties: 1) $f(z)$ is defined on $\overline{Z}$ and possesses the regular property X on E; 2) $|f(z)| \leqslant 1$; 3) $f(z)$ is analytic on $\overline{Z} \setminus E$; 4) $f(\infty) = 0$ (cf. the definition of $\mathfrak{M}(\overline{C}G, \infty)$ in [5, 7]).

It is easily shown that the class $M(X,E)$ consists of the single function $f(z) \equiv 0$ if and only if $\gamma_X(E) = 0$. Let us also note here that, in the case where X is the property $\mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$, $\gamma_X(E) = 0$ if and only if the inner Hausdorff measure of order $1+\alpha$ of this set E is zero: $\mathrm{mes}^{1+\alpha} E = 0$. In the case where X is the property of continuity, we shall call the X-capacity the AC-capacity.

3. Some theorems on approximation by rational fractions on domains of class $B(\alpha)$. Theorem 3. *Let G be a domain with finite perimeter such that every function $f(z)$, analytic inside G and possessing a regular property X on G, can be uniformly*** approximated on $\overline{G}$ by rational fractions. Then the X-capacity of the set $\Lambda(G)$ is equal to zero.*

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* $f(z)$ $(z \in E)$ is called uniformly differentiable on E if the ratio $[f(z)-f(\zeta)]/(z-\zeta)$, as $z \to \zeta$ $(z, \zeta \in E)$, tends to its limit $f'_E(\zeta)$ uniformly with respect to $\zeta \in E$. Obviously, a function $f(z)$ uniformly differentiable on E has at non-isolated points of this set a continuous derivative (on the set E).

** That is, analytic in some neighborhood of each point $z \notin E$.

*** The words "with arbitrary prescribed accuracy" are omitted here and below.

Theorem 4. In order that every function $f(z)$, analytic inside a domain $G \in B(1/3)$ and possessing on $\overline{G}$ a regular property X, be uniformly approximable

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on $\overline{G}$ by rational fractions, it is necessary and sufficient that the set $\Lambda(G)$ have zero X-capacity.

Corollary 1. In order that, on a domain $G \in B(1/3)$, every function analytic inside G and satisfying the condition $\operatorname{Lip}\alpha$ $(0 < \alpha < 1)$ on $\overline{G}$ be uniformly approximable by rational fractions, it is necessary and sufficient that

$$\operatorname{mes}^{1+\alpha}\Lambda(G) = 0.$$

Theorem 4′. In order that every function $f(z)$, analytic inside a domain $G \in B(1/2)$ and continuous on $\overline{G}$, be uniformly approximable on $\overline{G}$ by rational fractions, it is necessary and sufficient that the AC-capacity of the set $\Lambda(G)$ be equal to zero.

Theorem 5. Let $G \in B(1/3)$. A function $f(z)$, analytic inside $\overline{G}$ and continuous on $\overline{G}$, is uniformly approximable on $\overline{G}$ by rational fractions if and only if it is uniformly differentiable on $\Lambda(G)$.

We note that, in this case, $f(z)$ (uniformly approximable by rational fractions on $\overline{G}$) may fail to have a derivative at any point $z \in \Lambda(G)$, if the derivative is taken with respect to $\overline{G}$ [(6)]. From Theorem 4 and a generalization of Theorem 5 we obtain the following "removal of singularities" theorem:

Theorem 6. Let $G \in B(1/(m+2))$ (m natural) and $\gamma_X(\Lambda(G)) = 0$. Then every function $f(z)$, analytic inside G and possessing on $\overline{G}$ a (regular) property X, is m times uniformly differentiable on $\Lambda(G)$.

Corollary 1. Let $G \in B(1/(m+2))$ $(m \geq 1)$ and let $f(z)$ be continuous on $\overline{G}$ and analytic inside G. Then, if the AC-capacity of the set $\Lambda(G)$ is equal to zero, $f(z)$ is m times uniformly differentiable on $\Lambda(G)$.

Corollary 2. If $G \in B(1/(m+2))$ and $\operatorname{mes}^{1+\alpha}\Lambda(G) = 0$, then every function $f(z)$ satisfying on $\overline{G}$ the condition $\operatorname{Lip}\alpha$ $(0 < \alpha \leq 1)$ and analytic inside G is m times uniformly differentiable on $\Lambda(G)$.

In conclusion we note that theorems analogous to those stated above also hold for approximations in the metric $L_p(G)$ for $p \geq 2$. Thus, for example, there exist a simply connected domain G (as always, coinciding with the set of interior points of its closure $\overline{G}$) and a function $f(z)$, analytic in G and continuous on $\overline{G}$, such that

$$\iint_G |f(z) - R(z)|^p \, d\sigma \geq k = \text{const} > 0$$

for every rational $R(z)$ and every $p \geq 2$. It is interesting to compare this with a result of S. O. Sinanyan (reported at the Fourth All-Union Mathematical Congress), which consists in the following: let $p \geq 1$ and $\overline{G}$ be finitely connected; then every $f \in L_p(G)$, analytic in G, is approximable with arbitrary accuracy in the metric $L_p(G)$ by rational fractions**.

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* For $p = 2$ this result was obtained by Keldysh.

Note: Figure translations are in progress. See original paper for figures.

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