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Abstract

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ON THE ASYMPTOTIC VALUE OF THE APPROXIMATION OF MULTIPLY DIFFERENTIABLE FUNCTIONS BY LINEAR POSITIVE OPERATORS

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Many works have been devoted to the study of the asymptotic value of the approximation of differentiable functions by various linear polynomial operators. However, these works do not fully reveal the regularity that exists in the general case. In this note we establish the asymptotic dependence of the approximation of functions by arbitrary linear positive operators on derivatives of various orders of the functions being approximated.

Let $\mathfrak{M}(2m,x)=\{f(t)\}$ be the set of functions $f(t)$ defined on the interval $\langle a,b\rangle$ and having derivatives up to order $2m$ inclusive at the point $t=x\in\langle a,b\rangle$. Put

$$\Phi(t)=f(t)-\sum_{k=0}^{2m-1}\frac{f^{(k)}(x)}{k!}(t-x)^k.$$

Denote by $\mathfrak{M}_{\gamma_{2m}}(2m,x)$ the totality of all functions $f(t)$ from the set $\mathfrak{M}(2m,x)$ for which

$$\left|\frac{\Phi(t)-\Phi(x)}{\gamma_{2m}(t)}\right|\le M_\delta<+\infty$$

for all $|t-x|\ge\delta>0$ from the interval $\langle a,b\rangle$, where $\gamma_{2m}(t)=(t-x)^{2m}$ and δ is any fixed positive number.

Consider a family $\{\mathcal{L}_\lambda(f;x)\}$ of linear positive operators (abbreviated l.p.o.), defined on the set $\mathfrak{M}(2m,x)$, and in what follows we use the usual notation

$$\mathcal{L}_\lambda(f;x)=\mathcal{L}_\lambda[f(t);x].$$

In what follows we shall be interested in the asymptotic value of the approximation of functions $f(x)$ by means of the l.p.o. $\mathcal{L}_\lambda(f;x)$ as the parameter λ tends to some limiting value λ_0. Without loss of generality, assume that $\lambda_0=\infty$.

Put

$$\tau_\lambda^{[k]} = \mathcal{L}_\lambda[(t-x)^k; x] \qquad (k = 0, 1, 2, ...),$$

and let

$$\mathcal{L}_\lambda(1; x) = 1.$$

We shall prove the following theorem for functions from the set $\mathfrak{M}_{\gamma_{2m}}(2m, x)$ on any finite interval $\langle a, b\rangle$.

Theorem 1. In order that for all $f, \psi \in \mathfrak{M}_{\gamma_{2m}}(2m, x)$ the equality*

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* $f^{(2m)}(x)$ and $\psi^{(2m)}(x)$ do not vanish simultaneously. If $\psi^{(2m)}(x) = 0$, the right-hand side of (1) is taken to be equal to ∞.

$$\lim_{\lambda\to\infty} \frac{\mathcal{L}_\lambda(f; x) - f(x) - \sum_{k=1}^{2m-1} \frac{f^{(k)}(x)}{k!}\tau_\lambda^{[k]}}{\mathcal{L}_\lambda(\psi; x) - \psi(x) - \sum_{k=1}^{2m-1} \frac{\psi^{(k)}(x)}{k!}\tau_\lambda^{[k]}} = \frac{f^{(2m)}(x)}{\psi^{(2m)}(x)}, \tag{1}$$

it is necessary and sufficient that the condition

$$\lim_{\lambda\to\infty} \frac{\tau_\lambda^{[2m+2j]}}{\tau_\lambda^{[2m]}} = 0 \tag{2}$$

hold for at least one value of j $(j = 1, 2, ...)$.

Necessity. Consider the functions $\gamma_{2m}(t) = (t-x)^{2m}$ and $\gamma_{2m+2j}(t) = (t-x)^{2m+2j}$ $(j = 1, 2, ...)$. If (1) holds, then for these functions we find

$$\lim_{\lambda\to\infty} \frac{\tau_\lambda^{[2m+2j]}}{\tau_\lambda^{[2m]}} = \lim_{\lambda\to\infty} \frac{\mathcal{L}_\lambda(\gamma_{2m+2j}; x)}{\mathcal{L}_\lambda(\gamma_{2m}; x)} = \frac{\gamma_{2m+2j}^{(2m)}(x)}{\gamma_{2m}^{(2m)}(x)} = 0,$$

i.e. (2) is valid.

Sufficiency. Take the function

$$\Phi(t) = f(t) - \sum_{k=0}^{2m-1} \frac{f^{(k)}(x)}{k!}(t-x)^k.$$

It is obvious that

$$\lim_{t\to x}\frac{\Phi(t)-\Phi(x)}{\gamma_{2m}(t)}=\lim_{t\to x}\frac{f^{(2m-1)}(t)-f^{(2m-1)}(x)}{(2m)!(t-x)}=\frac{f^{(2m)}(x)}{(2m)!}. \tag{3}$$

Let

$$v_\delta(t)=\begin{cases}0, & |t-x|<\delta,\\ 1, & |t-x|\geq\delta\end{cases}$$

for arbitrary $\delta>0$. Then

$$v_\delta(t)(t-x)^{2m}\leq v_\delta(t)\frac{(t-x)^{2m+2j}}{\delta^{2j}}\leq\frac{(t-x)^{2m+2j}}{\delta^{2j}},\qquad j\geq 1.$$

Hence, taking into account the monotonicity of the linear operators $\mathcal{L}_\lambda(f;x)$, we have

$$\alpha_\lambda(\delta)=\mathcal{L}_\lambda[v_\delta(t)(t-x)^{2m};x]\leq\frac{\tau_\lambda^{[2m+2j]}}{\delta^{2j}}.$$

Consequently, by virtue of (2), we obtain

$$\lim_{\lambda\to\infty}\frac{\alpha_\lambda(\delta)}{\mathcal{L}_\lambda(\gamma_{2m};x)}=0 \tag{4}$$

for arbitrary $\delta>0$.

Now, on the basis of (4), from (3) one may conclude (see (3), p. 111) that the relation

$$\lim_{\lambda\to\infty}\frac{\mathcal{L}_\lambda(\Phi;x)-\Phi(x)}{\mathcal{L}_\lambda(\gamma_{2m};x)}=\frac{f^{(2m)}(x)}{(2m)!}$$

holds, or

$$\lim_{\lambda\to\infty}\frac{\mathcal{L}_\lambda(f;x)-f(x)-\sum_{k=1}^{2m-1}\frac{f^{(k)}(x)}{k!}\tau_\lambda^{[k]}}{\mathcal{L}_\lambda(\gamma_{2m};x)}=\frac{f^{(2m)}(x)}{(2m)!}.$$

From this (1) follows immediately.

Let us note that Theorem 1 admits a further generalization for a certain class of unbounded functions on any finite or infinite interval $\langle a,b\rangle$.

As an example, let us consider the polynomial of S. N. Bernstein

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}.$$

Since

$$B_n(1;x) = 1, \qquad \tau_n^{[1]} = B_n[(t-x);x] = 0,$$

$$\tau_n^{[2]} = B_n[(t-x)^2;x] = \frac{x(1-x)}{n},$$

$$\tau_n^{[4]} = B_n[(t-x)^4;x] = \frac{3n^2x^2(1-x)^2 + nx(1-x)(1-6x+6x^2)}{n^4},$$

the condition

$$\lim_{n\to\infty} \frac{\tau_n^{[4]}}{\tau_n^{[2]}} = 0$$

is satisfied. Hence, by Theorem 1,

$$\lim_{n\to\infty} \frac{B_n(f;x) - f(x)}{B_n[(t-y)^2;x]} = \frac{f''(x)}{2}$$

or

$$\lim_{n\to\infty} n\left[B_n(f;x) - f(x)\right] = \frac{x(1-x)}{2} f''(x). \tag{5}$$

Relation (5) was proved by E. V. Voronovskaya (^{1}).

In the same way, the general results of S. N. Bernstein (^{2}) are derived from Theorem 1.

Let us now consider a family $\{W_\lambda\}(f;x)$ of l.p.o. defined on the set $\overline{\mathfrak{M}}(2m,x)$ of 2π-periodic functions $f(t)$. For $W_\lambda(f;x)$ there is a theorem analogous to Theorem 1.

Let

$$\mu_\lambda^{[k]} = \mathcal{L}_\lambda\left[2^k \sin^k \frac{t-x}{2};x\right] \qquad (k=0,1,2,\ldots).$$

For functions of the set $\overline{\mathfrak{M}}(2,x)$ the following holds.

Theorem 2. In order that, for all $f, \psi \in \overline{\mathfrak{M}}(2, x)$, the equality

$$\lim_{\lambda\to\infty} \frac{W_\lambda(f;x) - f(x) - f'(x)\mu_\lambda^{[1]}}{W_\lambda(\psi;x) - \psi(x) - \psi'(x)\mu_\lambda^{[1]}} = \frac{f''(x)}{\psi''(x)} \tag{6}$$

hold, it is necessary and sufficient that the condition

$$\lim_{\lambda\to\infty} \frac{\mu_\lambda^{[2+2j]}}{\mu_\lambda^{[2]}} = 0 \tag{7}$$

be satisfied for at least one value j $(j = 1, 2, ...)$.

The proof is analogous to the proof of Theorem 1.

Let $q_k(t) = 2^k \sin^k \frac{t-x}{2}$. The necessity of the condition of the theorem follows from the equality

$$\lim_{\lambda\to\infty} \frac{\mu_\lambda^{[2+2j]}}{\mu_\lambda^{[2]}} = \lim_{\lambda\to\infty} \frac{W_\lambda(q_{2+2j};x)}{W_\lambda(q_2;x)} = \frac{q''_{2+2j}(x)}{q''_2(x)} = 0.$$

To prove sufficiency, consider the function

$$\varphi(t) = f(t) - f(x) - 2f'(x)\sin\frac{t-x}{2} \qquad (-\pi \leqslant t \leqslant \pi).$$

It is not difficult to prove that

$$\lim_{t\to x} \frac{\varphi(t) - \varphi(x)}{q_2(t)} = \frac{f''(x)}{2}.$$

Hence, on the basis of the equality

$$\lim_{\lambda\to\infty} \frac{\alpha_\lambda(\delta)}{\mu_\lambda^{[2]}} = 0,$$

which follows from (7), it follows that

$$\lim_{\lambda\to\infty} \frac{W_\lambda(f;x) - f(x) - f'(x)\mu_\lambda^{[1]}}{W_\lambda(q_2;x)} = \frac{f''(x)}{2}.$$

Let us note that Theorem 2 admits a further generalization analogous to Theorem 1.

From Theorem 2 there follow, in particular, some results of the author [7]. In addition, if $\lambda = n$ is a positive integer and

$$W_\lambda(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left[\frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt\right] dt = \mathcal{L}_n(f;x),$$

then from Theorem 2 there also follows the theorem of P. P. Korovkin [5].

Let

$$V_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left[\frac{1}{2} + \sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!} \cos kt\right] dt$$

be the singular Vallée-Poussin integral. In this case

$$\mu_n^{[2]} = \frac{2}{n+1}, \qquad \mu_n^{[4]} = \frac{2}{n+1}\left(4 - \frac{4n+2}{n+2}\right).$$

Consequently, from the condition

$$\lim_{n\to\infty} \frac{\mu_n^{[4]}}{\mu_n^{[2]}} = \lim_{n\to\infty}\left(4 - \frac{4n+2}{n+2}\right) = 0$$

it follows that

$$\lim_{n\to\infty} \frac{V_n(f;x) - f(x)}{V_n(\psi;x) - \psi(x)} = \frac{f''(x)}{\psi''(x)}.$$

Hence, for $\psi(t) = 4\sin^2 \frac{t-x}{2}$ we have

$$\lim_{n\to\infty} (n+1)\left[V_n(f;x) - f(x)\right] = f''(x),$$

i.e. we obtain the theorems of I. P. Natanson [4].

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CITED LITERATURE

[1] E. V. Voronovskaya, DAN (A), 79 (1932).
[2] S. N. Bernstein, DAN (A), 79 (1932).
[3] P. P. Korovkin, *Linear Operators and Approximation Theory*, Moscow, 1959.
[4] I. P. Natanson, DAN, 45, No. 7, 290 (1944).
[5] P. P. Korovkin, UMN, 13, issue 6 (1958).
[6] R. G. Mamedov, DAN, 128, No. 3 (1959).
[7] R. G. Mamedov, Izv. AN AzerbSSR, ser. phys., math., techn., No. 6 (1961).

Note: Figure translations are in progress. See original paper for figures.