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Abstract

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GENERAL THEORY OF THE MULTIPLICATION OF INDEPENDENT RANDOM VARIABLES

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0. The question of constructing a general theory of multiplication of independent random variables was posed by Paul Lévy (^{1,2}) as an independent problem, in connection with the fact that the scheme of multiplication of random variables (the M-scheme), being an essential extension of the scheme of summation (the A-scheme), cannot in general be reduced to the latter.

In the M-scheme a new operation on distribution functions (d.f.' s) appears, which we shall call M-composition and denote by $\circ$. Namely, if F_ξ, F_η are the d.f.' s of independent random variables ξ, η, and $\zeta = \xi\eta$, then $F_\zeta(x) = F_\xi(x) \circ F_\eta(x)$. This operation is commutative, associative, and distributive with respect to ordinary addition.

In view of the obvious analogy between the M- and A-schemes, it is natural to extend the terminology of the A-scheme to the M-scheme, accompanying terms and concepts with the prefix M for the M-scheme and the prefix A for the A-scheme, and to begin the construction of a general theory in the following direction.

1) Construction of an analytic apparatus playing in the M-scheme the same role as characteristic functions (c.f.' s) in the A-scheme.
2) Description in the M-scheme of the class $\mathfrak{M}$ of infinitely divisible laws (M-i.d.l.). By definition, a d.f. $F(x) \in \mathfrak{M}$ if there exists for it such a sequence of integers $0 < n_1 < n_2 < \cdots$ that F is representable in the form of an n_i-fold M-composition of some d.f. F_{n_i} (i.e. $F = F_{n_i} \circ F_{n_i} \circ \cdots \circ F_{n_i}$). If, moreover, the sequence $\{n_i\}$ contains an infinite subsequence of even numbers, then we assign the d.f. F to the subclass $\mathfrak{M}'$ of the class $\mathfrak{M}$ (unlike the A-scheme, $\mathfrak{M}'$ does not coincide with $\mathfrak{M}$).
3) Construction in the M-scheme of general limit theorems.
4) Clarification of the arithmetic properties of distributions with respect to the operation of M-composition.

This is the program to which we shall mainly adhere in what follows.

1. We associate with the random variable ξ the following pair of functions, putting here $0^{it} = 0$ (t takes real values):

$$w_k(t) = M|\xi|^{it} \operatorname{sgn}^k \xi, \qquad k = 0, 1.$$

Definition. The **characteristic transform** (c.t.) $W_\xi(t)$ of the random variable ξ will mean the diagonal matrix of second order with elements $\{w_0(t), w_1(t)\}$ on the main diagonal.

For matrices with diagonal elements $\{w_0(t), -w_1(t)\}$ and $\{\overline{w_0(t)}, \overline{w_1(t)}\}$ (complex conjugates) we shall adopt respectively the notations $W_\xi^*(t)$ and $\overline{W}_\xi(t)$. Let us note the most important properties of c.t.' s.

I. If the random variables ξ, η are independent, then

$$W_{\xi\cdot\eta}(t) = W_\xi(t) \cdot W_\eta(t).$$

II.

$$W_{-\xi}(t) = W_\xi^*(t); \qquad W_{1/\xi}(t) = W_\xi(-t) = \overline{W_\xi(t)}.$$

III.

$$c^+ = P\{\xi > 0\} = \tfrac{1}{2} \operatorname{sp} W_\xi(0), \qquad c^- = P\{\xi < 0\} = \tfrac{1}{2} \operatorname{sp} W_\xi^*(0).$$

IV. The functions $w_k(t)$ admit a unique representation of the form

$$w_k(t) = c^+ f^+(t) + (-1)^k c^- f^-(t),$$

where f^+, f^- are certain ch.f.' s defined by the equalities

$$c^+ f^+(t) = \frac{1}{2} \operatorname{sp} W_\xi(t), \qquad c^- f^-(t) = \frac{1}{2} \operatorname{sp} W_\xi^*(t).$$

Conversely, any pair of ch.f.' s f^+, f^-, together with a pair of nonnegative numbers c^+, c^- whose sum does not exceed 1, determine some ch.p. $W_\xi(t)$. Let us note that applying Pólya' s sufficient criterion to the construction of the ch.f.' s f^+ and f^- gives us a sufficient condition for a ch.p.

V. For all positive points of continuity of the d.f. $F_\xi(x)$, the following equality is valid (the case of negative values of x is reduced to this by means of property II):

$$F_\xi(t) = 1 - \frac{1}{2}c^+ - \frac{1}{2\pi} \int_0^\infty \operatorname{Im}\{x^{-it} \operatorname{sp} W_\xi(t)\} \frac{dt}{t}.$$

VI. The functions $w_k(t)$ determine some ch.p. $W(t)$ if and only if the functions $u(t) = \frac{1}{2}\operatorname{sp} W(t)$ and $v(t) = \frac{1}{2}\operatorname{sp} W^*(t)$: a) are continuous, $u(0) \geq 0$, $v(0) \geq 0$, $u(0) + v(0) \leq 1$; b) are positive definite.

VII. If a sequence of ch.p.'s $W_n(t)$ converges elementwise to a matrix $W(t)$ with continuous elements, then $W(t)$ is a ch.p. and the d.f.' s $F_n(x)$, determined by the ch.p.' s $W_n(t)$, converge weakly ($\Rightarrow$) to the d.f. $F(x)$ determined by the ch.p. $W(t)$. Conversely, if $W_n(t), W(t)$ are ch.p.' s for the d.f.' s $F_n(x)$ and $F(x)$, then from $F_n(x) \Rightarrow F(x)$ as $n \to \infty$ there follows uniform, in every finite interval, elementwise convergence of $W_n(t)$ to $W(t)$.

VIII. The role of moments with respect to a random variable ξ in the M-scheme is played by the quantities $m_k(n) = M(\log^n |\xi| \operatorname{sgn}^k \xi \mid \xi \neq 0)$, $n \geq 0$; $k = 0, 1$. The formal expansion of the ch.p. $W_\xi(t)$ has the form

$$W_\xi(t) = (c^+ + c^-) \sum_{n=0}^{\infty} \frac{(it)^n}{n!} M_n, \quad \text{where} \quad M_n = \begin{pmatrix} m_0(n) & 0 \\ 0 & m_1(n) \end{pmatrix}.$$

IX. Let a, b be arbitrary positive numbers and $\eta = a|\xi|^b \operatorname{sgn} \xi$. Then $\xi = |\eta/a|^{1/b} \operatorname{sgn} \eta$, $F_\eta(x) = F_\xi(|x/a|^{1/b} \operatorname{sgn} x)$, and $W_\eta(t) = a^{it} W_\xi(bt)$.

2. A description of the class $\mathfrak{M}$ (M-i.d. d.l.) was partially carried out by Lévy (^{1}) in terms of ch.f.' s of logarithms of random variables. A complete description of the class $\mathfrak{M}$ is given by the following theorem.

Theorem 1. The d.f. F belongs to $\mathfrak{M}$ if and only if the ch.p. corresponding to it has the form

$$w_0(t) = \alpha_0 f_1(t) f_2(t), \qquad w_1(t) = \alpha_1 f_1(t)/f_2(t),$$

where: a) $f_1(t)$ is the ch.f. of some A-i.d. d.l., $f_2(t)$ is the ch.f. of an A-i.d. d.l. of the form

$$\log f_2(t) = \int (e^{itu} - 1)\, dH(u);$$

b) $0 \leq \alpha_0 \leq 1$; $|\alpha_1| \leq \alpha_0 \lim_{\varepsilon \to 0} \exp\{2[H(\varepsilon) - H(-\varepsilon)]\}$. Membership in $\mathfrak{M}'$ is ensured by the additional (necessary and sufficient) condition $\alpha_1 \geq 0$.

We shall call a d.f. $F(x)$ M-stable if for any $a_i > 0$, $b_i > 0$, $i = 1, 2, 3$, there exist such $a > 0$, $b > 0$ that

$$F(a_1|x|^{b_1} \operatorname{sgn} x) \circ F(a_2|x|^{b_2} \operatorname{sgn} x) \circ F(a_3|x|^{b_3} \operatorname{sgn} x) = F(a|x|^b \operatorname{sgn} x).$$

Between M-stable (class $\mathfrak{M}$) and A-stable laws there exists a direct relationship, so that every A-stable law with density $g_A(x)$ determines in the following way exactly three M-stable laws with densities $g_M^{(1)}(x)$, $g_M^{(2)}(x)$, and $g_M^{(3)}(x)$:

$$g_M^{(1)}(x) = \begin{cases} 0, & x < 0, \\ g_A(\log x), & x \geq 0; \end{cases}$$

$$g_M^{(2)}(x) = \frac{1}{2} g_A(\log |x|); \qquad g_M^{(3)}(x) = g_M^{(1)}(-x).$$

If in the definition of the class $\mathfrak{N}$ of M-stable laws ($\mathfrak{N} \subset \mathfrak{M}$) one takes not three but two components in the left-hand side, then, in contrast to the A-scheme, we obtain a narrower class $\mathfrak{N}'$ ($\mathfrak{N}' \subset \mathfrak{N}$), consisting of laws of the form $g_M^{(1)}, g_M^{(2)}$. Let us note that an even number of components in the left-hand side of the definition leads to $\mathfrak{N}'$, and an odd number to $\mathfrak{N}$.

Analogues of the L-laws of A-schemes are distinguished in the class $\mathfrak{M}$, if in Theorem 1, as defining characteristic functions f_1, f_2, one chooses the characteristic functions of the A-infinitely divisible laws of the class L. The role of the Poisson law in the M-scheme is played by the distribution

$$P\{\xi = (-1)^r h^n\} = \frac{e^{-\lambda}}{2n!} \left[\alpha_0 e^{-\mu}(\lambda + \mu)^n + (-1)^r \alpha_1 e^{\mu}(\lambda - \mu)^n\right]$$

$$P\{\xi = 0\} = 1 - \alpha_0, \qquad r = 0, 1; \qquad n = 0, 1, 2, \ldots$$

where $0 \leq \alpha_0 \leq 1,\ |\alpha_1| \leq \alpha_0 e^{-2\mu};\ h, \lambda, \mu > 0$.

3. The formulation of limit theorems in the M-scheme is the same as in the A-scheme. A sequence of series of independent, within the series, random variables $\xi_{n1}, \ldots, \xi_{nk_n}$, $n = 1, 2, \ldots$, and the products

$$\zeta_n = a_n \xi_{n1} \xi_{n2} \cdots \xi_{nk_n}, \tag{1}$$

are considered, where a_n are some positive constants. It is required to give a description of the class of possible limiting distributions and convergence conditions for the products (1) under the assumption of "limit negligibility" of the random factors (M-negligibility): for every $\varepsilon > 0$

$$\lim_{n\to\infty} \max_k P\{(|\xi_{nk}| - 1)^2 > \varepsilon\} = 0. \tag{2}$$

In the scheme of increasing products $\zeta_n = \xi_1 \xi_2 \cdots \xi_n$, such M-negligibility is attained by means of a power transformation of the factors: $\bar{\zeta}_n = a_n |\zeta_n|^{b_n} \operatorname{sgn} \zeta_n$, where a_n and b_n are positive constants. Such a transformation is convenient and natural within the framework of the M-scheme by virtue of property IX.

Theorem 2. *The class of limiting distribution laws for products of the form (1), (2) coincides with the class* $\mathfrak{M}$.

Let $F_{nk}(x)$ denote the distribution functions of the random variables ξ_{nk},

$$c_{nk}^{+} = 1 - F_{nk}(+0), \qquad c_{nk}^{-} = F_{nk}(0);$$

$$\widetilde{F}_{nk}^{(1)}(x) = 1 - \frac{1}{c_{nk}^{+}}\left[1 - F_{nk}(e^{x})\right], \qquad \widetilde{F}_{nk}^{(2)}(x) = 1 - \frac{1}{c_{nk}^{-}} F_{nk}(-e^{x}),$$

and let $\tilde{\xi}_{nk}^{(\nu)}$, $\nu = 1, 2$, $k = 1, \ldots, k_n$, be a sequence of series of mutually independent random variables with distribution functions $\widetilde{F}_{nk}^{(\nu)}(x)$.

Theorem 3. *For convergence of the distribution functions of the products (1), (2), under some choice of constants $a_n > 0$, to a law of the class $\mathfrak{M}$, determined by $\alpha_0, \alpha_1, f_1(t)$ and $f_2(t)$, it is necessary and sufficient that the following conditions be fulfilled:*

In the case $\alpha_1 \neq 0$,

a)

$$\prod_{k=1}^{k_n} \left(c_{nk}^{+} + (-1)^{r} c_{nk}^{-}\right) \to \alpha_r \quad \text{as } n \to \infty, \quad r = 0, 1.$$

b) The sums $\tilde{\xi}_{n1}^{(\nu)} + \cdots + \tilde{\xi}_{nk_n}^{(\nu)} + \tilde{a}_n^{(\nu)}$, for some choice of constants $\tilde{a}_n^{(\nu)}$, have limiting laws with characteristic functions $f_\nu(t)$, $\nu = 1, 2$, and as the constants a_n one may take the quantities $\exp(\tilde{a}_n^{(1)} + \tilde{a}_n^{(2)})$.

In the case $\alpha_1 = 0$, condition a) remains, and b) is replaced by:

b_1) The sums $\tilde{\xi}_{n1}^{(1)} + \tilde{\xi}_{n1}^{(2)} + \cdots + \tilde{\xi}_{nk_n}^{(1)} + \tilde{\xi}_{nk_n}^{(2)} + \tilde{a}_n$, for some choice of constants $\tilde{a}_n$, have a limiting distribution law with characteristic function $f_1(t) f_2(t)$ (here one may take $\exp(\tilde{a}_n)$ as a_n).

4. The arithmetic properties of distributions with respect to the operation of M-composition are basically preserved, although peculiarities also arise.

Theorem 4. A ch. f. having no M-indecomposable components is M-infinitely divisible.

Theorem 5. If one of the components of the ch. f. $W(t)$ vanishes somewhere on the real axis, without being identically zero, then the ch. f. corresponding to this ch. f. has an M-indecomposable component. If, however, $w_1(t) \equiv 0^*$ (i.e., the distribution is symmetric), then the ch. f. admits a decomposition containing an M-indecomposable component.

Theorem 6. Every ch. f. is representable in the form of an M-composition of M-indecomposable ch. f.' s in a finite or countable number, and of an M-i. d. ch. f.

The peculiarities of the arithmetic of ch. f.' s in the M-scheme may be seen in the following assertions.

A nonsymmetric M-Poisson distribution law decomposes into an M-composition only of M-Poisson laws, whereas a symmetric one can also decompose into laws that are not M-Poisson. An identical assertion holds for M-normal laws.

5. It is of interest to describe and study distributions of the class $\mathfrak{F} = \mathfrak{G} \cap \mathfrak{M}$ (the $\mathfrak{G}$-class of A-i. d. laws). The fact that such a class is essentially nonempty is shown by the following theorem.

Theorem 7. All A-stable laws of the class W^{**} belong to $\mathfrak{M}$; moreover, the subclass $\mathfrak{M}'$ contains the laws having a nonnegative median.

6. Let ξ, η be independent random variables, $\zeta = \xi\eta$, and let ${}_{\xi}w_k(t)$, ${}_{\eta}w_k(t)$ be elements of the ch. f.' s corresponding to them. Then the condition ${}_{\xi}w_1(t) \cdot {}_{\eta}w_1(t) \equiv 0$ is necessary and sufficient for the symmetry of the ch. f. $F_{\zeta}(x)$. This is the answer to the question posed by Lévy in ([1]). Property IV makes it possible rather simply to construct examples of nonsymmetric ch. f.' s whose M-composition is symmetric.

Consider one-vertex (with vertex at zero) and two-vertex (with extrema at the points $-1, 0, 1$) distributions.

Theorem 8. A ch. f. $F_{\xi}(x)$ is one-vertex if and only if the matrix $(1 + it)W_{\xi}(t)$ is some ch. f.

As a consequence, from this we obtain that the M-composition of two ch. f.' s, one of which is one-vertex (with vertex at zero), is again one-vertex.

Theorem 9. A ch. f. $F_{\xi}(x)$ is two-vertex if and only if the corresponding ch. f. $W_{\xi}(t)$ is representable in the form

$$W_{\xi}(t) = \frac{1}{t} \int_0^t W(u)\, du,$$

where $W(u)$ is some ch. f.

It follows from Theorem 9 that the M-composition of two-vertex ch. f.' s with real ch. f.' s is again two-vertex.

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[1] P. Lévy, Ann. Sci. de l' Ecole Norm., **76**, No. 1, 59 (1959).
[2] P. Lévy, C. R., **248**, No. 13, 1920 (1959).

[3] V. M. Zolotarev, Theory of Probability and Its Applications, 2, issue 4, 444 (1957).

* The case $w_0(t) \equiv 0$ corresponds to an improper distribution.
** The definition of the class W is contained in ([3]), p. 453.

Note: Figure translations are in progress. See original paper for figures.

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