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Abstract

Full Text

MATHEMATICS

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ON SERIES OF RATIONAL FUNCTIONS

(Presented by Academician A. N. Kolmogorov, December 8, 1961)

1. It is known ^(1,2) that a sequence of polynomials $p_n(z)$ converging on a continuum E at the rate of a geometric progression (q^n , $0 < q < 1$) converges uniformly inside some domain containing the continuum E and depending on E and q (Walsh ⁽²⁾ calls this property the **overconvergence** of the sequence $p_n(z)$). An analogous picture holds in the case of sequences of rational functions $r_n(z)$ converging at the rate of a geometric progression, with one or another restriction on the location of the poles; in particular, when the poles of the rational functions have no limit points in some fixed domain containing ⁽²⁾ or not containing ⁽³⁾ the continuum E ; in the first case, as in the case of polynomials, overconvergence leads to analytic continuation of the limiting function, and in the second—to continuation in the well-known sense ⁽³⁾ as a quasianalytic one.

If no restrictions are imposed on the poles of the rational functions, then it is easy to construct sequences of rational functions converging on a continuum at the rate of a geometric progression and diverging at every point outside the continuum (cf. Remark 1 below). Thus, for example, the sequence

$$r_n(z) = \left(q \frac{1 - \bar{\alpha}_n z}{z - \alpha_n} \right)^n, \quad 0 < q < 1,$$

where $\alpha_1, \alpha_2, \dots$ is a set everywhere dense outside the closed unit disk, converges in the disk $|z| \leq 1$ to zero at the rate q^n and diverges for any z , $|z| > 1$. Nevertheless, if the sequence $r_n(z)$ converges on E sufficiently rapidly, then overconvergence takes place everywhere in the plane, except for a certain exceptional set representing the “cluster set” of the set of poles of the rational functions.

In the present note we give theorems establishing the dependence between the rate of convergence of rational functions (without any restrictions on the location of the poles) on E and the metric properties of this exceptional set (Theorems 1 and 2); in this case overconvergence leads to a function whose properties are analogous to those of functions monogenic in the sense of Borel ⁽⁴⁾ (Theorem 3).

2. It will be more convenient for us to consider not a sequence, but a series of rational functions

$$\sum_{n=0}^{\infty} r_n(z), \quad r_n(z) = \frac{a_{n0}z^n + \dots + a_{nn}}{(z - \alpha_{n1}) \dots (z - \alpha_{nn})}, \quad (1)$$

uniformly convergent on an arbitrary continuum E to a function $f(z)$ (all results are easily reformulated also for the case of a sequence $r_n(z)$). The Hausdorff measure of a set F corresponding to a function $h(r)$ will be denoted by $m_h(F)$.

Theorem 1. If

$$|r_n(z)| \leq \left(\frac{1}{n^{\alpha+\varepsilon}} \right)^n, \quad z \in E, \quad (2)$$

where $\alpha > 0$, $\varepsilon > 0$ is arbitrarily small, then the series (1) converges absolutely everywhere in the plane, with the exception of a set F , whose Hausdorff measure of order α^{-1} is equal to zero:

$$m_h(F) = 0, \quad h(r) = r^{1/\alpha}.$$

Moreover, whatever $\alpha > 0$, $q > 0$ may be, there exist series of the form (1) which diverge on a set of infinite Hausdorff measure of order α^{-1} (indeed, for every z from this set $|r_{n_k}(z)| \rightarrow \infty$, where n_k depends on z) and such that

$$|r_n(z)| \leq \left(\frac{q}{n^\alpha} \right)^n, \quad z \in E.$$

Remark 1. Theorem 1 is meaningful for $\alpha \geq 1/2$; in the limiting case $\alpha = 1/2$ the theorem guarantees overconvergence everywhere, except for a set of plane measure zero; if in condition (2) one omits ε (and still more if $\alpha < 1/2$), then overconvergence cannot be guaranteed even at a single point outside E . Moreover, for any $q > 0$ there exists a sequence $r_n(z)$ such that $|r_n(z)| \leq (q/\sqrt{n})^n$, $z \in E$, and for any point $z \notin E$ there is a subsequence $n_1, n_2, \dots, n_k, \dots$ such that $|r_{n_k}(z)| \rightarrow \infty$, $n_k \rightarrow \infty$.

Theorem 2. If

$$|r_n(z)| \leq e^{-n^{1+\beta+\varepsilon}}, \quad z \in E, \quad (3)$$

where $0 < \beta \leq 1$, $\varepsilon > 0$ is arbitrarily small, then the series (1) converges absolutely everywhere in the plane, with the exception of a set F such that

$$m_h(F) = 0, \quad h(r) = \left(\log \frac{1}{r} \right)^{-1/\beta};$$

in particular ($\beta = 1$), if $|r_n(z)| \leq e^{-n^{2+\varepsilon}}$, $z \in E$, then the logarithmic measure of the set F of points of divergence of the series (1) is zero.

Moreover, whatever β , $0 < \beta \leq 1$, may be, there exist series of the form (1) which diverge on a set of infinite h -measure,

$$h(r) = \left(\log \frac{1}{r} \right)^{-1/\beta},$$

and such that

$$|r_n(z)| \leq e^{-n^{1+\beta}}, \quad z \in E.$$

Remark 2. Theorems 1 and 2 can be sharpened in various directions:

- 1) Theorem 1 remains valid if condition (2) is replaced by the condition

$$|r_n(z)| \leq \rho_n^{\alpha n}, \quad \rho_n > 0, \quad \sum_{n=0}^{\infty} \rho_n < \infty, \quad z \in E;$$

analogously, in Theorem 2 condition (3) can be replaced by the condition

$$|r_n(z)| \leq e^{-n\rho_n^{-\beta}}, \quad \rho_n > 0, \quad \sum_{n=0}^{\infty} \rho_n < \infty, \quad z \in E \quad \text{for } 0 < \beta < 1,$$

$$|r_n(z)| \leq e^{-n \ln n / \rho_n}, \quad z \in E \quad \text{for } \beta = 1.$$

- 2) If instead of the series (1) one considers the series $\sum_{m=0}^{\infty} r_{n_m}(z)$, where $r_{n_m}(z)$ is a rational function of order n_m , then condition (2) of Theorem 1 can be replaced by the condition

$$|r_{n_m}(z)| \leq e^{-(\alpha+\varepsilon)n_m \ln n_m}, \quad z \in E.$$

This refinement is essential for lacunary series (1) (cf. Theorem 6 from (3)); an analogous refinement is also possible for Theorem 2.

- 3) Theorems 1 and 2 remain valid in the case when E is an arbitrary set of positive harmonic capacity.
- 4) We have formulated the theorems on overconvergence for the principal cases of the Hausdorff h -measure (in the interval from plane measure to logarithmic measure); an analogous theorem can be formulated for the general case of an h -measure (we do not give this theorem in view of the cumbersomeness of its formulation).

3. Borel ⁽⁴⁾ constructed an example of a nowhere dense set of type F_σ such that functions monogenic on this set (as well as functions that are limits of sequences of rational functions uniformly convergent “inside” this set, cf. ⁽⁵⁾) possess a number of properties analogous to the most important properties of analytic functions. The following theorem shows that, under the conditions of Theorems 1 and 2, the sum of the series (1) possesses properties analogous to the properties of functions monogenic in the sense of Borel (we restrict ourselves to stating the theorem corresponding to Theorem 1 for $\alpha = 1$).

Theorem 3. Let

$$|r_n(z)| \leq \frac{1}{n^{(1+\varepsilon)n}}, \quad \varepsilon > 0, \quad z \in E.$$

Then, whatever the closed bounded domain $D \supset E$ and the number $\delta > 0$ may be, there exists a system of open disks $K_1, K_2, \dots, K_n, \dots$ with radii $r_1, r_2, \dots, r_n, \dots$, $\sum_1^\infty r_n \leq \delta$, such that the closed set

$$D_\delta = D \setminus \bigcup_n K_n$$

has the following properties:

- 1) $D_\delta \subset D'_\delta$, if $D \subset D'$; $D_\delta \subset D_{\delta'}$, if $\delta' < \delta$;
- 2) the series (1) converges uniformly on the set D_δ ;

$$F(z) = \sum_{n=0}^{\infty} r_n(z), \quad z \in D_\delta;$$

- 3) the function $F(z)$ is monogenic on D_δ ; moreover, $F(z)$ is infinitely differentiable on the set D_δ (with respect to D_δ), and

$$F^{(p)}(z) = \sum_{n=0}^{\infty} r_n^{(p)}(z), \quad p = 1, 2, \dots, \quad z \in D_\delta;$$

the convergence is uniform on D_δ for every p .

Remark 3. Theorems 1, 2, and 3 can be formulated in the form of converse theorems on best approximations of a function $f(z)$ on E by rational functions. But since in converse theorems we are interested first of all in the properties of the function $f(z)$ itself and, on the other hand, the sequence of poles of the rational functions of best approximation (even if the best approximation is sought in the class of all rational functions, without any restrictions on the location of the poles) cannot be an arbitrary sequence (the fact that this is a sequence of best approximations imposes certain restrictions on the behavior of $\{\alpha_{nk}\}$)

as n increases), the corresponding converse theorems will no longer be sharp. Assertions analogous to the assertions of Theorems 1, 2, and 3 (concerning the properties of the sequence of rational functions of best approximation and thereby the properties of $f(z)$) are apparently already valid for classes of functions for which $R_n(f, E) \leq q^n$, $0 < q < 1$ (in particular, for the quasi-analytic class R_E ^(6,7); cf. also Theorem 6 from ⁽³⁾).

4. The proofs of Theorems 1, 2, and 3 are based, in particular, on the following lemmas on estimates of growth and estimates of derivatives of rational functions.

Lemma 1. Let E be a bounded set of positive harmonic capacity (in particular, a continuum); let D be an arbitrary bounded domain, $D \supset E$; and let $r_n(z)$ be a rational function of order n such that

$$|r_n(z)| \leq M \quad \text{for } z \in E.$$

Then, for any $H > 0$, $\alpha > 0$, and β , $0 < \beta \leq 1$, in the plane of the complex variable z there exists a system of disks K_1, K_2, \dots, K_m , $m \leq n$, with radii r_1, r_2, \dots, r_m such that:

$$1) \quad \sum_{i=1}^m (r_i)^{1/\alpha} \leq H,$$

and for all $z \in D$ lying outside this system of disks the estimate

$$|r_n(z)| \leq M \left(\frac{C(\alpha; D)}{H^\alpha} \right)^n$$

holds, or

$$2) \quad \sum_{i=1}^m \left(\log \frac{1}{r_i} \right)^{-1/\beta} \leq H,$$

and

$$|r_n(z)| \leq M e^{C(\beta; D)n/H^\beta}, \quad 0 < \beta < 1,$$

$$|r_n(z)| \leq M e^{C(D)n \ln n/H}, \quad \beta = 1, \quad z \in D \setminus \bigcup K_i.$$

Lemma 2. Let $r_n(z)$ be an arbitrary rational function of order n (in particular, a polynomial of degree n); whatever $H > 0$, $\alpha > 0$, and β , $0 < \beta \leq 1$, may

be, in the complex plane there exists a system of disks K_1, K_2, \dots, K_m , $m \leq 2n$, with radii r_1, \dots, r_m such that:

$$1) \quad \sum_{i=1}^m (r_i)^{1/\alpha} \leq H,$$

and for all z lying outside this system of disks, the estimate

$$|r'_n(z)| \leq \frac{C(\alpha)\varphi(n)}{H^\alpha} |r_n(z)|$$

holds, where

$$\varphi(n) = \begin{cases} n^\alpha, & \text{for } \alpha > 1, \\ n \ln n, & \text{for } \alpha = 1, \\ n, & \text{for } \alpha < 1, \end{cases}$$

or

$$2) \quad \sum_{i=1}^m \left(\log \frac{1}{r_i} \right)^{-1/\beta},$$

and

$$|r'_n(z)| \leq |r_n(z)| e^{(1+\varepsilon)n^\beta/H^\beta}, \quad z \notin \bigcup K_i,$$

where $\varepsilon > 0$ is arbitrary, $r_i < r(\varepsilon, \beta)$.

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