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Abstract

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A. Kh. GUDIEV

AN EMBEDDING THEOREM FOR THE TRACE IN ABSTRACT FUNCTIONS

(Presented by Academician S. L. Sobolev on 16 VI 1962)

From the results of S. L. Sobolev (1), with additions by V. I. Kondrashov (3) and V. P. Il' in (4), the following theorem is known:

Theorem 1. If $f(\mathbf{x}) \in W_p^{(l)}(\Omega)$ and $n > lp$, then $f(\mathbf{x})$ is equivalent (up to a set of measure zero) to a certain function that has a well-defined trace $\tilde{f}$ on any hyperplane S_s of dimension $s > n - lp$, and on these hyperplanes $\tilde{f} \in L_q$, where $q \leq \frac{sp}{n - lp}$.

As shown in (2), this theorem cannot be strengthened in these terms, and therefore the result is final. It is natural to pose the question: what can be said about the trace $\tilde{f}$ of a function $f \in W_p^{(l)}(\Omega)$ on hyperplanes S_s of dimension $s \leq n - lp$. If one takes into account that the trace $\tilde{f}$ of the function $f(\mathbf{x})$ on the hyperplane S_s depends on the $(n-s)$-dimensional vector $\mathbf{x}_{n-s}$, i.e., is an abstract function of the vector $\mathbf{x}_{n-s}$ in $(n-s)$-dimensional space, then one can clarify the property of the trace on arbitrary hyperplanes, i.e., one can prove the following problem, posed by S. L. Sobolev and S. M. Nikol' skii at the Fourth All-Union Mathematical Congress (5).

Theorem 2. If $f(\mathbf{x}) \in W_p^{(l)}(\Omega)$,

$$\frac{n}{p} - l < \frac{n-s}{p_2} + \frac{s}{p_1},$$

where $p_2 \geq p_1 \geq p > 1$, then the trace $\tilde{f}$ of the function $f(\mathbf{x})$ on any hyperplane S_s of dimension $s \leq n$, as an abstract function, belongs to the Bochner space B_{p_2}, i.e.

$$\left\{ \int_{S_{n-s} \cap \Omega} \|\tilde{f}\|^{p_2}_{L_{p_1}(S_s \cap \Omega)} \, d\mathbf{x}_{n-s} \right\}^{1/p_2} < \infty.$$

In the present paper a solution of the stated problem and a certain generalization of it are given.

Let us consider the class $L_{(p_1,p_2)}(\Omega)$ of functions $f(\mathbf{x})$, defined on Ω, for which the norm

$$\|f\|_{L_{(p_1,p_2)}(\Omega)} = \left\{ \int_{S_{n-s}\cap\Omega} \left[\int_{S_s\cap\Omega} |f(\mathbf{x})|^{p_1}\, d\mathbf{x}_s \right]^{p_2/p_1} d\mathbf{x}_{n-s} \right\}^{1/p_2}$$

is bounded, and prove a theorem concerning properties of integrals of potential type.

Theorem 3. If $f(\mathbf{x}) \in L_p(\Omega)$; $\lambda < \frac{n-s}{p_2} + \frac{s}{p_1} + \frac{n}{p'}$; $p_2 \geq p_1 \geq p > 1$, then

$$u(\mathbf{x}) = \int_\Omega \frac{f(\mathbf{y})}{r^\lambda}\, d\mathbf{y} \in L_{(p_1,p_2)}(\Omega)$$

and, moreover,

$$\|u\|_{L_{(p_1,p_2)}(\Omega)} \leqslant c\|f\|_{L_p(\Omega)}, \tag{1}$$

where c is a constant independent of f, u.

Consider and estimate

$$|u(x)| \leqslant \int_\Omega |f(y)|\, r^{-\frac{n-s}{p_2} - \frac{s}{p_1} - \frac{n}{p'} + \varepsilon}\, dy =$$

$$= \int_\Omega \left(r^{-\frac{n}{p'} + \varepsilon_1}\right) \left(|f(y)|^{p\left(\frac{1}{p} - \frac{1}{p_1}\right)}\right) \left(|f(y)|^{\frac{p}{p_1}} r^{-\frac{n-s}{p_2} - \frac{s}{p_1} + \varepsilon_2}\right) dy$$

$$(\varepsilon_1 + \varepsilon_2 = \varepsilon).$$

Since

$$\frac{1}{p_1} + \left(\frac{1}{p} - \frac{1}{p_1}\right) + \frac{1}{p'} = 1,$$

putting

$$\lambda_1 = \frac{1}{p_1}; \qquad \lambda_2 = \frac{1}{p} - \frac{1}{p_1}; \qquad \lambda_3 = \frac{1}{p'}$$

and applying Hölder' s inequality to three factors, and then applying the generalized Minkowski inequality (^{6}), taking into account that $p_1 > 1$, after obvious transformations we obtain:

$$|u(x)|^{p_1} \leqslant C_1^{p_1} \|f\|_{L_p(\Omega)}^{p_1\left(1-\frac{p}{p_1}\right)} \int_\Omega |f(y)|^p r^{-\left(\frac{n-s}{p_2}+\frac{s}{p_1}-\varepsilon_2\right)p_1}\, dy. \tag{2}$$

Integrating both sides of inequality (2) over the hyperplane S_s and interchanging the order of integration on the right-hand side, we obtain

$$\int_{S_s\cap\Omega} |u(x)|^{p_1}\, dx_s \leqslant C_1^{p_1} \|f\|_{L_p(\Omega)}^{p_1\left(1-\frac{p}{p_1}\right)} \int_\Omega |f(y)|^p \left(\int_{S_s\cap\Omega} r^{-\frac{(n-s)p_1}{p_2}-s+\varepsilon_2 p_1}\, dx_s\right) dy. \tag{3}$$

Putting

$$r_s = \left[\sum_1^s (x_i-y_i)^2\right]^{1/2}; \qquad r_{n-s} = \left[\sum_{s+1}^n (x_i-y_i)^2\right]^{1/2}; \qquad \alpha_1 = \frac{(n-s)p_1}{p_2}$$

$$-\varepsilon_4 p_1; \qquad \alpha_2 = s - \varepsilon_3 p_1 \quad (\varepsilon_3+\varepsilon_4=\varepsilon_2)$$

and taking into account the inequality

$$(A^2+B^2)^{1/2(\lambda_1+\lambda_2)} \geqslant A^{\lambda_1} B^{\lambda_2},$$

valid for $\alpha_1, \alpha_2, A, B > 0$, we obtain from (3)

$$\int_{S_s\cap\Omega} |u(x)|^{p_1}\, dx_s \leqslant C_2 C_1^{p_1} \|f\|_{L_p(\Omega)}^{p_1\left(1-\frac{p}{p_1}\right)} \int_\Omega |f(y)|^p r^{-\frac{(n-s)p_1}{p_2}+\varepsilon_4 p_1}\, dy.$$

If $p_2 > p_1$, then we may apply the generalized Minkowski inequality to the estimate of the integral

$$\int_{S_{n-s}\cap\Omega} \left[\int_{S_s\cap\Omega} |u(x)|^{p_1}\, dx_s\right]^{\frac{p_2}{p_1}} dx_{n-s} \leqslant$$

$$\leqslant C_2^{\frac{p_2}{p_1}} C_1^{p_1} \|f\|_{L_p(\Omega)}^{p_2\left(1-\frac{p}{p_1}\right)} \int_{S_{n-s}\cap\Omega} \left(\int_\Omega |f(y)|^p r_{n-s}^{-\frac{(n-s)p_1}{p_2}+\varepsilon_4 p_1}\, dy\right)^{\frac{p_2}{p_1}} dx_{n-s} \leqslant$$

$$\leqslant C_2^{\frac{p_2}{p_1}} C_1^{p_1} \|f\|_{L_p(\Omega)}^{p_2\left(1-\frac{p}{p_1}\right)} \left[\int_\Omega |f(y)|^p \left(\int_{S_{n-s}\cap\Omega} r^{-(n-s)+\varepsilon_4 p_2}\, dx_{n-s}\right)^{\frac{p_1}{p_2}} dy\right]^{\frac{p_2}{p_1}} =$$

$$= C_3 C_2^{\frac{p_2}{p_1}} C_1^{p_1} \|f\|_{L_p(\Omega)}^{p_2\left(1-\frac{p}{p_1}\right)} \left(\int_\Omega |f(y)|^p \, dy\right)^{\frac{p_2}{p_1}} = C_4^{p_2} \|f\|_{L_p(\Omega)}^{p_2}. \tag{4}$$

If, however, $p_2 = p_1$, then, changing at once the order of integration, we obtain the same estimate. Inequality (1) follows from (4).

Theorem 4. If $f(x) \in W_p^{(l)}(\Omega)$ and

$$\frac{n}{p} - l < \frac{n-s}{p_2} + \frac{s}{p_1}, \qquad p_2 \geqslant p_1 \geqslant p > 1,$$

then

$$f(x) \in L_{(p_1,p_2)}(\Omega)$$

and, moreover,

$$\|f\|_{L_{(p_1,p_2)}(\Omega)} \leqslant c\|f\|_{W_p^{(l)}(\Omega)}. \tag{5}$$

We use S. L. Sobolev' s integral representation of functions $f(x) \in W_p^{(l)}(\Omega)$

$$f(x) = \sum_{|\alpha| \leq l-1} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \int_\Omega \xi_{\bar\alpha}(y) f(y) \, dy + \sum_{|\bar\alpha| = l} \int_\Omega \frac{\omega_{\bar\alpha}(x,y)}{r^{n-l}} D^{\bar\alpha} f(y) \, dy;$$

$$\|f\|_{L_{(p_1,p_2)}(\Omega)} \leqslant \left\{ \int_{S_{n-s}\cap\Omega} \left[\int_{S_s\cap\Omega} \left(\sum_{|\alpha|\leq l-1} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \int_\Omega \xi_{\bar\alpha}(y) f(y)\, dy \right)^{p_1} dx_s \right]^{\frac{p_2}{p_1}} dx_{n-s} \right\}^{\frac{1}{p_2}}$$

$$+ \sum_{|\bar\alpha|=l} \left\{ \int_{S_{n-s}\cap\Omega} \left[\int_{S_s\cap\Omega} \left(\int_\Omega \frac{\omega_{\bar\alpha}(x,y)}{r^{n-l}} D^{\bar\alpha} f(y)\, dy \right)^{p_1} dx_s \right]^{\frac{p_2}{p_1}} dx_{n-s} \right\}^{\frac{1}{p_2}} \equiv I_1 + \sum_{|\bar\alpha|=l} I_{\bar\alpha}. \tag{6}$$

By the hypothesis of the theorem,

$$\frac{n}{p} - l < \frac{n-s}{p_2} + \frac{s}{p_1},$$

whence

$$n - l < \frac{n-s}{p_2} + \frac{s}{p_1} + \frac{n}{p'}. \tag{7}$$

Since $D^{\bar\alpha} f(y) \in L_p(\Omega)$ and inequality (7) holds, on the basis of Theorem 3 we obtain

$$\left\{ \int_{S_{n-s}\cap\Omega} \left[\int_{S_s\cap\Omega} \left(\int_\Omega \frac{\omega_{\bar\alpha}(x,y)}{r^{n-l}} D^{\bar\alpha} f(y)\, dy \right)^{p_1} dx_s \right]^{\frac{p_2}{p_1}} dx_{n-s} \right\}^{\frac{1}{p_2}} \leqslant C_1 \|f\|_{W_p^{(l)}(\Omega)}. \tag{8}$$

An analogous estimate is obtained for I_1:

$$I_1 \leqslant C_2 \|f\|_{W_p^{(l)}(\Omega)}. \tag{9}$$

From (6), (8), (9) follows (5).
The validity of Theorem 2 follows from Theorem 4.

Denote each point $x \in R_n$ by

$$x = (x_{s_1}, x_{s_2}, \dots, x_{s_k}),$$

where

$$\sum_1^k s_i = n;$$

$$x_{s_1}(x_1, \dots, x_s), \quad x_{s_2}(x_{s_1+1}, \dots, x_{s_1+s_2}), \dots, x_{s_k}(x_{s_1+s_2+\dots+s_{k-1}+1}, \dots, x_n).$$

Consider the set of functions $f(x)$, defined on $\Omega \in R_n$, for which the norm

$$\|f\|_{L_{(p_1,p_2,\dots,p_k)}(\Omega)} = \left(\int_{S_{s_k}}\left(\int_{S_{s_{k-1}}}\cdots\left(\int_{S_{s_2}}\left(\int_{S_{s_1}}|f(x)|^{p_1}\,dx_{s_1}\right)^{\frac{p_2}{p_1}}dx_{s_2}\right)^{\frac{p_3}{p_2}}\cdots dx_{s_{k-1}}\right)^{\frac{p_k}{p_{k-1}}}dx_{s_k}\right)^{\frac{1}{p_k}}$$

is bounded.

Theorem 5. If $f(x) \in L_p(\Omega)$, $\lambda < \sum_1^k \frac{s_i}{p_i} + \frac{n}{p'}$, where $1 < p \leq p_1 \leq p_2, \dots, p_k$, $\sum_1^k s_i = n$, then

$$u(x) = \int_\Omega \frac{f(y)}{r^\lambda}\,dy \in L_{(p_1,p_2,\dots,p_k)}(\Omega)$$

and, moreover,

$$\|u\|_{L_{(p_1,p_2,\dots,p_k)}(\Omega)} \leq C_3 \|f\|_{L_p(\Omega)}.$$

Theorem 6. If $f(x) \in W_p^{(l)}(\Omega)$, $\frac{n}{p} - l < \sum_1^k \frac{s_i}{p_i}$, where $1 < p \leq p_1 \leq p_2, p_3, \dots, p_k$, $\sum_1^k s_i = n$, then

$$f(x) \in L_{(p_1,p_2,\dots,p_k)}(\Omega)$$

and, moreover, the following holds:

$$\|f\|_{L_{(p_1,p_2,\dots,p_k)}(\Omega)} \leq C_4 \|f\|_{W_p^{(l)}(\Omega)}.$$

Theorem 7. Let the function $f(y_1,\dots,y_s,y_{s+1},\dots,y_n)$ be summable over the whole space of n variables with exponent $p > 1$, and let $\varphi(x_{s+1},\dots,x_n)$ be summable over the space of $n-s$ variables with exponent $p_2' > 1$, $s \leq n$.

Then the inequality holds

$$\left\{ \int_{R_x^s} \left[\int_{R_x^{n-s}} \int_{R_y^n} \frac{f(y_1,\dots,y_s,y_{s+1},\dots,y_n)\,\varphi(x_{s+1},\dots,x_n)}{r^\lambda}\, dx_{n-s}\, dy \right]^{p_1} dx_s \right\}^{\frac{1}{p_1}} \leq$$

$$\leq k \|f\|_{L_p} \|\varphi\|_{L_{p_2'}},$$

if

$$\lambda = \frac{n-s}{p_2} + \frac{s}{p_1} + \frac{n}{p'}, \qquad p_2 \geq p_1 > p > 1,$$

the constant k depends on p, p_1, p_2, s, n.

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Institute of Mathematics with Computing Center
of the Siberian Branch of the Academy of Sciences of the USSR

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