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Abstract

Full Text

MATHEMATICS

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OPTIMAL PROCESSES WITH A CYCLE CONSTRAINT

(Presented by Academician L. S. Pontryagin, 15 XII 1961)

Below we consider time-optimal problems ⁽¹⁾ for systems of difference equations. The constraints on the control actions have a specific character and are called by us cyclic.

1. Let the behavior of the control system be described by the equation

$$x(n + 1) = Ax(n) + bu(n), \tag{1}$$

where $x = \{x_1, x_2, \dots, x_l\}$ is an l -dimensional vector, $A = \{a_{ij}\}$ is a constant $l \times l$ matrix, $b = \{b_1, b_2, \dots, b_l\}$ is a constant vector, and $u(n)$ is the control function.

The optimal problems to be discussed below are as follows. Initial conditions $x(0) = \{x_1(0), x_2(0), \dots, x_l(0)\}$ are given. It is required to find a function $u(n)$ such that the point $x(n)$, moving along the trajectory of equation (1), reaches the origin in the least number of steps K^0 . In doing so, the control functions must satisfy one of the following constraints.

Given a number $\omega > 0$, define the number N from the condition $N\omega < K^0 \leq (N + 1)\omega$, and put $u(n) = 0$ for $K^0 \leq n \leq (N + 1)\omega$.

Problem I

$$\max_{0 \leq k \leq N} \sum_{i=k\omega}^{(k+1)\omega-1} |u_i| \leq 1. \tag{2}$$

Problem II

$$\sum_{k=0}^N \max_{k\omega \leq i \leq (k+1)\omega-1} |u_i| \leq 1. \tag{3}$$

Problem III

$$\max_{0 \leq k \leq N} \left(\sum_{i=k\omega}^{(k+1)\omega-1} |u_i|^p \right)^{1/p} \leq 1, \quad p > 1. \quad (4)$$

Problem IV.

$$\sum_{i=1}^{\omega-1} \max |u_{k\omega+i}| \leq 1. \quad (5)$$

Remark. We set $u_n = u(n)$.

Problems (2)–(5) differ from the usual problems of optimal control in that the constraints on the control functions are cyclic in character (the cycle length is ω steps). Each of the four problems can be given a physical interpretation. For example, the following physical problem is possible. A certain process is regulated by the quantity of fuel supplied. Fuel is supplied directly to the process from a certain bunker, which is periodically (after ω units of time) filled from another reservoir whose stocks are practically unlimited. Under such conditions it is necessary to specify the optimal fuel consumption. If we assume that the process under study can be described by an equation of type (1), then we obviously have Problem I.

The number of problems of type (2)–(5) can be significantly increased, but we shall confine ourselves to these in order to show the method of solving them, which without substantial changes can also be applied to other problems. Problems I–IV below are solved by reducing them to the L -problem ⁽²⁾ in functional spaces (in our case we restrict ourselves to finite-dimensional spaces).

2. In this section certain finite-dimensional spaces are introduced, and the spaces conjugate to them are determined. We shall need the following spaces.

- 1) The space (EI) . Its dimension is K^0 . The norm is introduced by the formula:

$$\|h\| = \sum_{k=0}^N \max_{k\omega \leq i \leq (k+1)\omega-1} |h_i|.$$

- 2) The space (EII) , whose elements are normed as follows:

$$\|h\| = \max_{0 \leq k \leq N} \sum_{i=k\omega}^{(k+1)\omega-1} |h_i|.$$

- 3) The space $(EIII)$, normed as follows:

$$\|h\| = \sum_{k=0}^N \left(\sum_{i=k\omega}^{(k+1)\omega-1} |h_i|^q \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

4) The space (EIV) , the norm in which is defined by the relation

$$\|h\| = \max_{0 \leq i \leq \omega-1} \sum_{k=0}^N |h_{k\omega+i}|.$$

Let us define the spaces conjugate to the spaces (EI) – (EIV) given above.

1) The space $(EI)^*$, the norm in which is defined as follows:

$$\|f\| = \max_{0 \leq k \leq N} \sum_{i=k\omega}^{(k+1)\omega-1} |f_i|.$$

2) The space $(EII)^*$, normed as follows:

$$\|f\| = \sum_{k=0}^N \max_{k\omega \leq i \leq (k+1)\omega-1} |f_i|.$$

3) The space $(EIII)^*$, whose elements are normed as follows:

$$\|f\| = \max_{0 \leq k \leq N} \left(\sum_{i=k\omega}^{(k+1)\omega-1} |f_i|^p \right)^{1/p}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

4) The space $(EIV)^*$. The norm is introduced by the formula:

$$\|f\| = \sum_{i=0}^{\omega-1} \max_{0 \leq k \leq N} |f_{k\omega+i}|.$$

3. Let us apply the known scheme ⁽³⁾ for reducing an optimal problem to an L -problem. We write the solution of the nonhomogeneous equation (1) by means of Cauchy' s formula

$$x(n) = F(n)x(0) + \sum_{s=0}^{n-1} F(n-s-1)bu(s), \quad (6)$$

where $F(n)$ is the fundamental matrix of solutions of the homogeneous equation (1) for $u(n) \equiv 0$.

By assumption, at the instant K^0 the trajectory reaches the origin of coordinates: $x(K^0) = 0$. Denote $h(s) = F(-s-1)b$; then (6) takes the form

$$-x_i(0) = \sum_{s=0}^{K^0-1} h_i(s)u(s), \quad i = 1, 2, \dots, l. \quad (7)$$

Putting $c'_i = -x_i(0)$, $u_i = f_i$, $h_i = h^i$, $m = l$, we evidently obtain the L -problem (2), defined on the spaces (EI) – (EIV) . The solvability condition (2,3) will have the following form:

$$\begin{aligned} \lambda(K^0) &= \min \|\xi_1 h_1 + \xi_2 h_2 + \dots + \xi_l h_l\| \geq 1, \\ \xi_1 x_1(0) + \xi_2 x_2(0) + \dots + \xi_l x_l(0) &= -1. \end{aligned} \quad (8)$$

Let $\psi = \xi_1^0 h_1 + \xi_2^0 h_2 + \dots + \xi_l^0 h_l$ be a solution of problem (8). Then the solution of problem (7) minimal in norm is sought (2) from the condition that it has the element ψ as extremal.

Problem I. The optimal control has the form

$$\begin{aligned} u(k\omega + \mu_i^k) &= C_k^i \operatorname{sing} [\xi_1^0 h_1(s) + \xi_2^0 h_2(s) + \dots + \xi_l^0 h_l(s)]_{s=k\omega + \mu_i^k}, \\ i &= 1, 2, \dots, q_k, \end{aligned}$$

the instants μ_i^k are determined from the condition

$$\begin{aligned} &\max_{0 \leq i < \omega-1} |[\xi_1^0 h_1(s) + \dots + \xi_l^0 h_l(s)]_{s=k\omega + i}| = \\ &= |[\xi_1^0 h_1(s) + \dots + \xi_l^0 h_l(s)]_{s=k\omega + \mu_1^k}| = \dots = |[\xi_1^0 h_1(s) + \dots + \xi_l^0 h_l(s)]_{s=k\omega + \mu_{q_k}^k}| > 0, \end{aligned}$$

the constants C_k^i satisfy the condition

$$\sum_{i=1}^{q_k} C_k^i = \frac{1}{\lambda(K^0)},$$

and the numbers $\xi_1^0, \xi_2^0, \dots, \xi_l^0$ are determined from the solution of the problem

$$\lambda(K^0) = \min \sum_{k=0}^N \max_{k\omega \leq i \leq (k+1)\omega-1} |\xi_1 h_1(i) + \xi_2 h_2(i) + \dots + \xi_l h_l(i)| \geq 1$$

under

$$\xi_1 x_1(0) + \xi_2 x_2(0) + \dots + \xi_l x_l(0) = -1.$$

At the remaining points $u(s) = 0$.

If for some \tilde{k}

$$\max_{0 \leq i < \omega - 1} |[\xi_1^0 h_1(s) + \dots + \xi_l^0 h_l(s)]_{s=\tilde{k}\omega+i}| = 0,$$

then for the points $\{\tilde{k}\omega + i\}$, $i = 0, 1, \dots, \omega - 1$, we can say nothing about the value of $u(j)$ at them, except for the condition

$$\sum_{j=\tilde{k}\omega}^{(k+1)\omega-1} |u_j| \leq \frac{1}{\lambda(K^0)}.$$

Problem II. The optimal control is given by the formula

$$u(k_\mu \omega + i) = C_{k_\mu} \operatorname{sign} [\xi_1^0 h_1(s) + \xi_2^0 h_2(s) + \dots + \xi_l^0 h_l(s)]_{s=k_\mu \omega+i},$$

$$\mu = 1, 2, \dots, q,$$

where the numbers k_μ are determined from the condition

$$\max_{0 \leq k \leq N} \sum_{i=k\omega}^{(k+1)\omega-1} |\psi_i| = \sum_{i=k_1\omega}^{(1+k_1)\omega-1} |\psi_i| = \dots = \sum_{i=k_q\omega}^{(k_q+1)\omega-1} |\psi_i|;$$

the coefficients C_{k_μ} are such that $\sum_{\mu=1}^q C_{k_\mu} = \frac{1}{\lambda(K^0)}$. At points where $\psi_i = \xi_1^0 h_1(i) + \xi_2^0 h_2(i) + \dots + \xi_l^0 h_l(i) = 0$, the values $u(i)$ are not defined. The numbers $\xi_1^0, \xi_2^0, \dots, \xi_l^0$ are the solution of the problem

$$\min_{0 \leq k \leq N} \max_{i=k\omega}^{(k+1)\omega-1} |\xi_1 h_1(i) + \xi_2 h_2(i) + \dots + \xi_l h_l(i)| = \lambda(K^0) \geq 1$$

under

$$\xi_1 x_1(0) + \xi_2 x_2(0) + \dots + \xi_l x_l(0) = -1.$$

Problem III. The optimal control is as follows:

$$u(k\omega + i) = \frac{[\xi_1^0 h_1(s) + \xi_2^0 h_2(s) + \dots + \xi_l^0 h_l(s)]_{s=k\omega+i}}{\lambda(K^0) \left(\sum_{i=k\omega}^{(k+1)\omega-1} |\psi_i|^p \right)^{1/p}} \operatorname{sign} \psi_{k\omega+i},$$

where the numbers $\xi_1^0, \xi_2^0, \dots, \xi_l^0$ are the solution of the problem

$$\lambda(K^0) = \min \sum_{k=0}^N \left(\sum_{i=k\omega}^{(k+1)\omega-1} |\xi_1 h_1(i) + \xi_2 h_2(i) + \dots + \xi_l h_l(i)|^q \right)^{1/q} \geq 1$$

under

$$\xi_1 x_1(0) + \xi_2 x_2(0) + \dots + \xi_l x_l(0) = -1, \quad 1/p + 1/q = 1.$$

If $\psi_i \equiv 0$, $k\omega \leq i \leq (k+1)\omega - 1$, then the control at these points is determined from additional conditions.

Problem IV. The optimal control is sought in the form

$$u(k\omega + i_\mu) = C_{i_\mu} \operatorname{sign}[\xi_1^0 h_1(s) + \xi_2^0 h_2(s) + \dots + \xi_l^0 h_l(s)]_{s=k\omega+i_\mu},$$

where the instants i_μ , $\mu = 1, 2, \dots, q$, are determined as follows:

$$\max_{0 \leq i \leq \omega-1} \sum_{k=0}^N |\psi_{k\omega+i}| = \sum_{k=0}^N |\psi_{k\omega+i_1}| = \dots = \sum_{k=0}^N |\psi_{k\omega+i_q}|,$$

the numbers C_{i_μ} are such that $\sum_{\mu=1}^q C_{i_\mu} = \frac{1}{\lambda(K^0)}$, and the numbers $\xi_1^0, \xi_2^0, \dots, \xi_l^0$ are the solution of the problem

$$\lambda(K^0) = \min \max_{0 \leq i \leq \omega-1} \sum_{k=0}^N |[\xi_1 h_1(s) + \xi_2 h_2(s) + \dots + \xi_l h_l(s)]_{s=k\omega+i}| \geq 1$$

under

$$\xi_1 x_1(0) + \xi_2 x_2(0) + \dots + \xi_l x_l(0) = -1.$$

The value of the control at points where $\psi_i = 0$ is determined from other conditions.

Remark. The control at points about which nothing has been said is equal to zero.

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