Reports of the Academy of Sciences of the USSR

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1962

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1962. Volume 145, No. 6

MATHEMATICAL PHYSICS

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HADAMARD' S MULTIPLICATION THEOREM AND THE ANALYTIC CONTINUATION OF THE TWO-PARTICLE UNITARITY CONDITION

(Presented by Academician N. N. Bogolyubov on 11 IV 1962)

Hadamard is due a profound theorem according to which, between the singularities of three functions defined by power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$
$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (1)$$
$$h(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

there is the following multiplicative law of composition:

$$\gamma = \alpha\beta, \quad (2)$$

where α, β, γ are the singularities, respectively, of the functions $f(z), g(z), h(z)$. A rigorous formulation of this theorem is given in ([1]).

In mathematical physics, an important role is played by series in Legendre polynomials

$$F(z) = \sum_{n=0}^{\infty} a_n P_n(z), \quad (3)$$

which, for functions $F(z)$ analytic on the interval $[-1, 1]$, converge in ellipses with foci at the points ± 1.

It is very interesting that between the singularities β of the function $F(z)$, defined by the series (3), and the singularities α of the function $f(z)$, defined by the Taylor series with the same coefficients

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{4}$$

there is the following simple dependence, established by Faber ([2]):

$$\beta = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right). \tag{5}$$

Combining Hadamard' s theorem with Faber' s theorem, we immediately obtain the following theorem, found recently by Gunson and Taylor ([3]).

Let $a(z)$ and $b(z)$ be two functions regular on the interval $[-1, 1]$ such that their expansions in Legendre polynomials

$$\begin{aligned} a(z) &= \sum_{n=0}^{\infty} a_n P_n(z), \\ b(z) &= \sum_{n=0}^{\infty} b_n P_n(z) \end{aligned} \tag{6}$$

converge inside the maximal ellipses E_a and E_b with foci ± 1, inside which these functions are analytic. Let $\{\alpha\}$ and $\{\beta\}$ denote the sets of singular and exterior points of the functions $a(z)$ and $b(z)$. Then the function

$$f(z) = \sum_{n=1}^{\infty} a_n b_n P_n(z) \tag{7}$$

is regular on the segment $[-1, 1]$ and has an analytic continuation along any finite path that does not intersect any point of the set

$$-\{\alpha\beta + (\alpha^2 - 1)^{1/2}(\beta^2 - 1)^{1/2}\}$$

and does not return to the segment $[-1, 1]$.

In other words, the singular and exterior points of the function $f(z)$ can lie only on the set of points of the form

$$\alpha\beta + (\alpha^2 - 1)^{1/2}(\beta^2 - 1)^{1/2}, \tag{8}$$

where at the outset the principal values of the roots are taken.

The circumstance that transformation (5) makes it possible to reduce the study of analytic properties of series in Legendre polynomials to power series has, in our opinion, important significance, since there is an extensive literature on the theory of Taylor series ([1]).

Below we indicate one application of the theorem of Gunson—Taylor to the study of the analytic properties of the scattering amplitude.

Let $T(W^2, \cos\theta)$ be the scattering amplitude of two scalar particles, where W is the total invariant energy, and θ is the scattering angle in the center-of-mass system. As Lehmann showed ([4]), from the conditions of spectrality, Lorentz invariance, and causality there follows the analyticity of $T(W^2, \cos\theta)$ in the variable $z = \cos\theta$ in the "small Lehmann ellipse," i.e., the possibility of the expansion

$$T(W^2, \cos\theta) = 8\pi \frac{W}{K} \sum_{l=0}^{\infty} (2l+1) T_l(W^2) P_l(\cos\theta),$$

$$W^2 = 4(K^2 + m^2). \tag{9}$$

S. Mandelstam ([5]), in a number of works, studied the question of how the unitarity requirement affects the analytic properties of the amplitude; he formulated this requirement in the two-particle approximation. One of his results may be formulated as follows. Let T_{ie}, T_{io} be, respectively, the transition amplitudes between the initial and intermediate two-particle state and between the intermediate and final state. Let $z_{ie} = \cos\theta_{ie}$, $z_{io} = \cos\theta_{io}$ be the singular points of the amplitudes T_{ie}, T_{io}; $z = \cos\theta$ the singular points of the imaginary part $\operatorname{Im} T(W^2, \cos\theta)$, determined from the two-particle unitarity condition.

Then the points z can be found by the following composition law:

$$z = z_{ie} z_{io} + (z_{ie}^2 - 1)^{1/2} (z_{io}^2 - 1)^{1/2}. \tag{10}$$

Comparison of formulas (8) and (10) shows their identity. And indeed, Mandelstam' s result follows immediately from the multiplication theorem of Gunson —Taylor after Mandelstam' s two-particle unitarity condition is written in the known way through partial amplitudes

$$\operatorname{Im} T_l(W^2) = T_{l(ie)}^* T_{l(io)}. \tag{11}$$

It seems to us that the remarks made in the present communication may play some role in further study of the analytic properties of the scattering amplitude.

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Received
11 IV 1962

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Note: Figure translations are in progress. See original paper for figures.