

Academician S. L. SOBOLEV

Theorem 1.** *Let a system of functions be given

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Abstract

Full Text

MATHEMATICS

Academician S. L. SOBOLEV

ON THE NUMBER OF NODES OF CUBATURE FORMULAS ON THE SPHERE

In previous notes ^(1,2) we considered cubature formulas on the sphere that converge in order of proximity in the space of all possible series in spherical functions. In the present note we attempt to estimate asymptotically the gain thereby obtained.

Theorem 1. *Let a system of functions be given*

$$\psi_1(x), \psi_2(x), \dots, \psi_K(x) \quad (1)$$

with integrals

$$\iint \psi_j(x) d\Omega = b_j. \quad (2)$$

In order that the formula

$$(l, f) \equiv \iint_{\Omega} f d\Omega - \sum_{k=1}^N C_k f(x^{(k)}) = 0 \quad (3)$$

with a prescribed system of nodes can be made exact for all $f = \psi$ by a suitable choice of the coefficients C_k , it is necessary and sufficient that this formula be exact for all such linear combinations

$$f(x) = a_1\psi_1 + a_2\psi_2 + \dots + a_K\psi_K, \quad (4)$$

which vanish at the nodes $x^{(k)}$.

This theorem establishes a duality between problems of interpolation and numerical integration.

The proof follows from comparing two systems of equations:

$$\sum_{k=1}^N C_k \psi_j(x^{(k)}) \equiv \overline{AC} = \vec{b} \quad (5)$$

for finding the coefficients C_k of the cubature formula (3), and

$$\sum_{j=1}^K a_j \psi_j(x^{(k)}) = \bar{a}A = 0 \quad (6)$$

for finding the coefficients a_1, a_2, \dots, a_K in the representation (4). Here A denotes the matrix

$$A = (\psi_j(x^{(K)})).$$

Remark. If the rank of the matrix A is equal to K , then the system (6) has no solutions, and then it is always possible to choose the coefficients C_k so that formula (3) is valid for all ψ_j .

Theorem 2. Consider some sequence of systems of nodes

$$x_r^{(1)}, x_r^{(2)}, \dots, x_r^{N(r)}, \quad r = 1, 2, \dots, \quad (7)$$

for cubature formulas of the form (3).

In order that cubature formulas (3), exact for all functions from some finite set (1) of analytic functions ψ_j , should correspond to all these systems, beginning with some r , $r > R$, it is sufficient that the following condition be satisfied. There exists a domain $\Omega_0 \subset \Omega$ such that the nodes (7) form an ε -net for any ε , beginning with some $r > r(\varepsilon)$.

The idea of the proof consists in considering the determinants of order K of the matrix A . These determinants are values of the function Δ of K variables:

$$\Delta(x^{(1)}, x^{(2)}, \dots, x^{(K)}) = \begin{vmatrix} \psi_1(x^{(1)}) & \dots & \psi_1(x^{(K)}) \\ \vdots & \ddots & \vdots \\ \psi_K(x^{(1)}) & \dots & \psi_K(x^{(K)}) \end{vmatrix} \quad (8)$$

for different particular values $x^{(1)}, x^{(2)}, \dots, x^{(K)}$.

In the domain $\Omega_0 \times \Omega_0 \times \dots \times \Omega_0$ there exist points where this determinant is nonzero. Since $\Delta(x^{(1)}, \dots, x^{(K)})$ is an analytic function and the points (7) fall into an arbitrarily close neighborhood of any such point in the matrix, there will be nonzero determinants. By virtue of the preceding remark, theorem 2 follows from this.

Consideration of formulas symmetric with respect to a group of rotations, as we have seen, makes it possible to restrict oneself only to harmonics invariant with respect to the same group of rotations. To satisfy all conditions (3), we then have at our disposal only the coefficients C_k at nonequivalent nodes, the number L of which is greater than N/M , where N is the total number of points

and M is the order of the group. The number $\sigma(n)$ or $\sigma^*(n^*)$ of symmetric harmonics up to the given order inclusive then proves to grow more slowly than $L(n)$. Therefore, estimating approximately,

$$L(N) = \sigma(n) \quad \text{or} \quad L(N) = \sigma^*(n). \quad (9)$$

Fig. 1

Fig. 1

Fig. 2

Fig. 2

Without invoking considerations of symmetry, generally speaking, with maximal use of all the parameters at our disposal, we could obtain

$$(n + 1)^2 = 3N; \quad (10)$$

$(n + 1)^2$ is the number of all spherical harmonics up to order n inclusive; $3N$ is the number of free parameters in formula (3). (Apart from the coefficient C_k , each point $x^{(k)}$ is itself determined by two parameters.) Formulas invariant with respect to the icosahedral group give precisely such a gain for small N . For large N the gain will be smaller. It is convenient to estimate the gain by comparing the functions $N(L)$ and $n(\sigma)$ or $n^*(\sigma^*)$.

Let us compute $N(L)$ for two lattices obtained by projecting onto the sphere triangular nets arranged symmetrically on each face of invariant polyhedra (see Figs. 1 and 2).

For the first type, with the number of nodes on the side of a triangle equal to k , one obtains for the full rotation group, putting $k = 6s + r$, $r < 6$:

$$N = \frac{M}{6}k^2 + 2;$$

$$\begin{aligned} L &= (s + 1)(k - 3s) && \text{for } r > 0, \\ L &= (s + 1)(k - 3s) + 1 && \text{for } r = 0, \end{aligned} \quad (11)$$

where M is the order of the group of proper rotations of the polyhedron, equal to half the full order of the group.

For the second type, with $k = 2s + r$, $r < 2$:

$$N = \frac{M}{2}k^2 + 2;$$

$$\begin{aligned} L &= (s+1)^2 && \text{for } r = 0; \\ L &= (s+1)(s+2) && \text{for } r = 1. \end{aligned} \quad (12)$$

Hence we obtain, for example, for $k = 6s + 5$ in the first case,

$$N(L) = 2M \left[L - \sqrt{3L^{1/2}} + \dots \right], \quad (13)$$

and for $k = 2s + 1$ in the second case:

$$N(L) = 2M \left[L - 2L^{1/2} + \dots \right]. \quad (14)$$

Computing σ and σ^* for

$$n = KM/2 - 1 \quad (15)$$

as was indicated in the preceding note, we obtain

$$\sigma \left(K \frac{M}{2} - 1 \right) = K \sigma \left(\frac{M}{2} - 1 \right) + \frac{K(K-1)M}{2}; \quad (16)$$

$$\sigma^* \left(K \frac{M}{2} - 1 \right) = K \sigma^* \left(\frac{M}{2} - 1 \right) + \frac{K(K-1)M}{4}; \quad (17)$$

$$\sigma \left(\frac{M}{2} - 1 \right) = \frac{M}{4};$$

$$\sigma^* \left(\frac{M}{2} - 1 \right) = 5 \quad \text{for } G_{\text{VIII}},$$

$$\sigma^* \left(\frac{M}{2} - 1 \right) = 11 \quad \text{for } G_{\text{XX}}.$$

This gives

$$\sigma \left(K \frac{M}{2} - 1 \right) = \frac{K^2 M}{4}$$

and, consequently,

$$[n(\sigma) + 1]^2 = M\sigma, \quad (18)$$

and also

$$(n^* + 1)^2 = 2M \left[\sigma^* - \sqrt{\frac{49}{30}} \sigma^{*1/2} + \dots \right] = 2M [\sigma^* - 1.3\sigma^{*1/2} + \dots] \quad (19)$$

for the icosahedral groups, and

$$(n^* + 1)^2 = 2M \left[\sigma^* - \sqrt{\frac{4}{3}} \sigma^{*1/2} + \dots \right] = 2M [\sigma^* - 1.16\sigma^{*1/2} + \dots] \quad (20)$$

for the octahedral group.

Table 1

N	n	$(n+1)^2$	L	N	n	$(n+1)^2$	L	N	n	$(n+1)^2$	L	N	n	$(n+1)^2$	L
6	3	16	1	14	5	26	2	12	5	36	1	32	9	100	2
18	5	32	2	50	9	100	4	42	9	100	2	122	15	256	4
38	7	64	3	110	11	144	6	92	11	144	3	272	19	400	6
66	9	100	4	194	15	256	9	162	15	258	4	482	25	676	9
102	11	144	5					262	17	324	5				
146	13	196	7												
198	15	256	8												
258	17	324	10												

Comparison of (13) and (14) with (19) and (20) characterizes the gain obtained by means of symmetric formulas of the given type for large L . As is seen, this gain is small. Equating $L = \sigma^*$ for small values, we obtain a table of the functions $n(N)$, $[n(N) + 1]^2$, and $L(N)$ for the octahedron and icosahedron in formulas of the first and second type (see Table 1).

Formulas with 4 points for the icosahedron group were indicated in the work of V. A. Ditkin ⁽³⁾.

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Note: Figure translations are in progress. See original paper for figures.

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