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L. N. SHEVRIN and V. M. KOPYTOV

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Abstract

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L. N. SHEVRIN and V. M. KOPYTOV

SEMIGROUPS WITH RELATIVELY COMPLEMENTED SUBSEMIGROUPS

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By $\Sigma(\Gamma)$, as in ([1−3]), we denote the set of all subsemigroups of the semigroup Γ, partially ordered by inclusion, and by $\Sigma'(\Gamma)$ the enlarged set obtained by adjoining the empty set to $\Sigma(\Gamma)$. $\Sigma(\Gamma)$ is not always a lattice, since the intersection of two subsemigroups may turn out to be empty. $\Sigma'(\Gamma)$ is a complete lattice.

In the present note we describe the semigroups Γ for which $\Sigma(\Gamma)$ has relative complements, i.e. every closed interval $[A, B]$ is a complemented lattice. In other words, for any triple of subsemigroups A, H, B such that $A \subseteq H \subseteq B$, there exists a subsemigroup F such that $H \cap F = A$ and $\{H, F\} = B$ (by $\{H, F\}$ we denote the subsemigroup generated by the set $H \cup F$ of elements of the semigroup Γ; the symbols $\cap$ and $\cup$ denote the set-theoretic operations of intersection and union). By analogy with ([4]) we shall call such semigroups RK-semigroups*. It is obvious that the class of RK-semigroups contains the class of RK-groups, partially studied in the book ([4])**, and also the class of semigroups Γ for which $\Sigma'(\Gamma)$ is a lattice with relative complements. The latter class is described in ([6]).

Consider a finite semigroup $\{x\}$ generated by an element x. Let m be the least natural number such that $x^h = x^m$, where $h < m$. Put $d = m - h$. Then the pair of numbers (h, d) is called the **type** of the element x (see ([7])).

By K_e we shall denote the set of all elements of the given periodic semigroup some power of which is equal to the idempotent e of this semigroup.

A semigroup Γ is called a **bundle** (([8]), see also ([7])) of its subsemigroups $\Gamma_\alpha, \Gamma_\beta, \ldots$, called the **components** of the bundle, if all the components are pairwise disjoint, their set-theoretic union is equal to Γ, and for each pair $\Gamma_\alpha, \Gamma_\beta$ of these subsemigroups there is such a subsemigroup Γ_γ that $\Gamma_\alpha\Gamma_\beta \subseteq \Gamma_\gamma$. A bundle is called **matrix** ([7,8]) if each of its components can be supplied with a pair of indices $\Gamma_{\xi\eta}$, where ξ runs through all elements of some set X, and η through all elements of a set Y, and, moreover, for any $\xi_1, \xi_2 \in X$ and $\eta_1, \eta_2 \in Y$ one has

$$\Gamma_{\xi_1\eta_1}\Gamma_{\xi_2\eta_2} \subseteq \Gamma_{\xi_1\eta_2}.$$

A semigroup with zero is called a semigroup with **zero multiplication** if the product of any two of its elements is equal to zero.

§ 1. Properties of RK-semigroups.

In this section we shall consider a number of properties of RK-semigroups. The corresponding propositions are used—

* RK-groups in ([4]) are called groups for which the lattice of subgroups has relative complements. An arbitrary RK-group is periodic. If Γ is a periodic group, then $\Sigma(\Gamma)$ coincides with the lattice of all subgroups of Γ. Since every RK-semigroup will be periodic (Lemma 2), the intersection of the class of RK-semigroups with the class of groups gives exactly the class of RK-groups.

** Among works published after the book ([4]), let us note, for example, the paper ([5]), where, in particular, RK-groups are considered.

are used in the proof of the main theorem on RK-semigroups, to which the following section is devoted.

First of all, let us note the following obvious lemma.

Lemma 1. *A subsemigroup of an RK-semigroup is also an RK-semigroup.*

Lemma 2. *An arbitrary RK-semigroup is periodic, and each of its elements has type $(2,1)$ or $(1,n)$, where n is not divisible by the square of any prime.*

Corollary. *Let x be an arbitrary element of an RK-semigroup and let $x \in K_e$. Then one of the equalities holds*

$$xe = ex = e, \qquad xe = ex = x.$$

This corollary is used essentially, in particular, in the proof of the following important lemma.

Lemma 3. *The set of all idempotents of an arbitrary RK-semigroup is its subsemigroup.*

Lemma 4. *In order that a semigroup Γ, all of whose elements are idempotents, be an RK-semigroup, it is necessary and sufficient that $\Sigma'(\Gamma)$ be a structure with relative complements.*

We now state the main theorem of the paper ([6]).

Theorem 1. *In order that $\Sigma'(\Gamma)$ be a structure with relative complements, it is necessary and sufficient that all elements of the semigroup Γ be idempotents and that for any $x, y \in \Gamma$ the alternative*

$$xyx = x, \qquad xyx = y$$

hold.

From Lemmas 3, 1, 4 and Theorem 1 it follows that

Lemma 5. *If e, i are idempotents of an arbitrary RK-semigroup, then the alternative*

$$eie = e, \qquad eie = i. \tag{*}$$

holds.

Remark 1. The second of the equalities in condition $(*)$ is equivalent to

$$ei = ie = i.$$

Remark 2. If the idempotents e and i of an RK-semigroup are not permutable, then the first of the equalities in condition $(*)$ is valid, which means, as is easy to see, that one of the following conditions holds:

a) $ei = e, \quad ie = i$;

b) $ei = i, \quad ie = e$;

c) $e \neq ei \neq i, \quad e \neq ie \neq i, \quad eie = e, \quad iei = i.$

Let us consider RK-semigroups possessing a unique idempotent*. Their structure is described by the following theorem.

Theorem 2. *In order that a semigroup Γ with one idempotent e be an RK-semigroup, it is necessary and sufficient that in Γ there exist subsemigroups G and H such that $G \cup H = \Gamma$, $G \cap H = e$, G is an RK-group, H is a semigroup with zero multiplication, and for any $g \in G$, $h \in H$ one has $gh = hg = g$.*

Theorem 2 now makes it possible to pass to the consideration of RK-semigroups containing more than one idempotent.

Lemma 6. *If, in an RK-semigroup containing more than one idempotent, distinct idempotents e and i are permutable, then the classes K_e and K_i consist only of these idempotents.*

Lemma 7. *In an RK-semigroup containing more than one idempotent, there are no elements of type $(2, 1)$.*

Lemma 8. *An arbitrary RK-semigroup containing more than one idempotent is a bundle of RK-groups.*

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* It is not difficult to note that for such and only such semigroups $\Sigma(\Gamma)$ is a structure with relative complements.

Lemma 9. *If, in an RK-semigroup containing more than one idempotent, one of the components of the band is one-element, then all the remaining components are also one-element.*

§ 2. **Main theorem.** The results of the preceding paragraph make it possible to prove the following main theorem on RK-semigroups.

Theorem 3. *In order that a semigroup be an RK-semigroup, it is necessary and sufficient that it be a semigroup of one of the following types:*

1) *a semigroup described by Theorem 1;*

2) *a semigroup described by Theorem 2;*

3) *a matrix band of (isomorphic) RK-groups whose identities form a subsemigroup.*

We give a concise proof of the theorem.

Sufficiency. If a semigroup Γ satisfying the conditions of the theorem has type 1) or 2), then it will be an RK-semigroup, which follows from Theorems 1 and 2. Let Γ have type 3), and let $K_{\alpha\beta}$, $K_{\gamma\delta}$ be two arbitrary components of the band, whose corresponding identities are e and i (we do not assume isomorphism of the components; it will be obtained in the proof of necessity). From the definition of a matrix band we obtain $K_{\alpha\beta}K_{\gamma\delta}K_{\alpha\beta} \subseteq K_{\alpha\beta}$. Hence, and from the fact that the idempotents in Γ form a subsemigroup, it follows that $eie = e$. Thus the subsemigroup E of all idempotents of the semigroup Γ satisfies the conditions of Theorem 1; consequently, E will be an RK-semigroup.

Let A, H, B be arbitrary subsemigroups of Γ such that $A \subseteq H \subseteq B$. Denote $E \cap A = E_A$, $E \cap H = E_H$, $E \cap B = E_B$. Obviously, $E_A \subseteq E_H \subseteq E_B$. Then there exists a subsemigroup $E' \subseteq E$ such that $E_H \cap E' = E_A$ and $\{E_H, E'\} = E_B$. Take an arbitrary idempotent $e \in E_A$. Denote $A \cap K_e = A_e$, $H \cap K_e = H_e$, $B \cap K_e = B_e$. A_e, H_e, B_e are subsemigroups of the RK-group K_e, and $A_e \subseteq H_e \subseteq B_e$. Therefore there exists a subsemigroup $F_e \subseteq B$ such that $H_e \cap F_e = A_e$ and $\{H_e, F_e\} = B_e$. Put $F = \{F_e, E'\}$. One can show that $H \cap F = A$, $\{H, F\} = B$. Let us prove, for example, the second equality (the proof of the first is somewhat more complicated and takes more space).

Let b be an arbitrary element of B, and let $b \in K_i$. We have $i \in E_B$ and $ebe \in K_e \cap B = B_e$. Since $E_B = \{E_H, E'\}$ and $B_e = \{H_e, F_e\}$, it follows that $i \in \{E_H, E'\} \subseteq \{H, F\}$ and $ebe \in \{H_e, F_e\} \subseteq \{H, F\}$. Then $b = ibi = ieibei = i \cdot ebe \cdot i \in \{H, F\}$. Thus $B \subseteq \{H, F\}$. The reverse inclusion is obvious; therefore $\{H, F\} = B$.

Necessity. Let Γ be an RK-semigroup which is neither a semigroup of type 1) nor a semigroup of type 2). We shall show that it has type 3). By Lemma 8, Γ is a band of RK-groups, each component of which, by Lemma 9, consists of more than one element. Then from Lemma 6 it follows that Γ does not contain distinct permutable idempotents. Hence, by Remark 2 to Lemma 5, we obtain that for arbitrary idempotents $e, i \in \Gamma$ the equality $eie = e$ holds.

On the set E of all idempotents of the semigroup Γ, which by Lemma 3 is a subsemigroup, introduce the following binary relations ρ_1 and ρ_2:

$$e \sim i \ (\rho_1) \quad \text{if and only if} \quad ei = i;$$

$$e \sim i \ (\rho_2) \quad \text{if and only if} \quad ei = e.$$

Using Remark 2 to Lemma 5, one can show that each of these relations is an equivalence relation. The set E is then split into two families of equivalence classes with respect to the relations ρ_1 and ρ_2. To each class of the first family we assign some symbol, in such a way that distinct classes correspond to distinct symbols. Denote the set of all these symbols by X. We proceed analogously with the second family. We obtain a set of symbols Y. It is easy to see that the intersection of an arbitrary class of the first family with any class of the second family consists of only one element. Thus, to each

an idempotent from Γ, but also to assign to each component of the band a corresponding pair of symbols $\xi \in X$, $\eta \in Y$. Namely, for ξ we take the symbol corresponding to that equivalence class with respect to ρ_1 which contains this idempotent, and for η the symbol corresponding to the analogous class with respect to ρ_2. The component of the band to which the symbols ξ and η correspond will be denoted by $K_{\xi\eta}$.

Let $K_{\xi_1\eta_1}, K_{\xi_2\eta_2}$ be arbitrary components, and let $K_{\xi_1\eta_1}K_{\xi_2\eta_2} \subseteq K_{\xi\eta}$. Considering the relations between the idempotents of these components, it is not hard to verify that $\xi = \xi_1$, $\eta = \eta_2$. This means precisely that Γ is a matrix band of its components. The isomorphism of the components follows from the fact that a semigroup of this structure is a completely simple semigroup without zero. Every completely simple semigroup without zero is a set-theoretic sum of pairwise disjoint isomorphic groups (see (7)).

Remark 1. Types 1) and 3) of RK-semigroups containing more than one idempotent are not mutually exclusive; namely, the intersection of the corresponding classes consists of matrix bands of one-element semigroups.

Remark 2. Every RK-semigroup of type 3) belongs to the class of semigroups all of whose subsemigroups coincide with their idealizers. This class was considered in (9).

Remark 3. The description of RK-semigroups given by Theorem 3, in types 2) and 3), has essentially been reduced to the description of RK-groups. The structure of arbitrary RK-groups has apparently not yet been completely studied, although under some additional restrictions complete descriptions of the corresponding types of groups are known (see, for example, (4, 5)). In particular, finite RK-groups have been studied.

Remark 4. The isomorphism of the components of the matrix band for a semigroup of type 3) in Theorem 3 could also have been obtained without considering completely simple semigroups, since the following theorem is valid, whose conditions are satisfied in our case.

If the components of a matrix band of semigroups possess identities and these identities form a subsemigroup, then the components are isomorphic.

Ural State University
named after A. M. Gorky

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Note: Figure translations are in progress. See original paper for figures.