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Abstract

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AN ERGODIC THEOREM FOR RANDOM FIELDS HOMOGENEOUS IN THE BROAD SENSE

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In the theory of random processes an important role is played by Neumann’ s ergodic theorem, according to which, for any continuous random process $\xi(t)$, $-\infty < t < +\infty$, stationary in the broad sense, there exists a shift-invariant mean value

$$\bar{\xi} = \underset{T\to+\infty}{\mathrm{l.\,i.\,m.}} \frac{1}{2T}\int_{-T}^{T} \xi(t)\,dt$$

and $D\bar{\xi} < \infty$. Wiener $^{(18)}$ generalized this theorem to homogeneous random fields on m-dimensional Euclidean space R^m; further generalizations were given in the works of Pitt $^{(15)}$ and Calderon $^{(10)}$. The present note is devoted to extending Neumann’ s theorem to homogeneous (in the broad sense) generalized random fields on R^m (Sec. 1) and to homogeneous (in the broad sense) continuous random fields on groups (Sec. 2).

1. Let $I(\varphi)$ be a homogeneous generalized random field over the space K of infinitely differentiable finite functions $\varphi(\mathbf{x})$, $\mathbf{x} = (x_1, \dots, x_m)$; $F(d\vec{\lambda})$ and $\Phi(d\vec{\lambda})$ are respectively the mean-square and random spectral measures of this field, where

$$\int_{R^m} \frac{F(d\vec{\lambda})}{(1+\lambda^2)^p} < +\infty$$

for some integer $p \geq 0$ ($\vec{\lambda} = (\lambda_1, \dots, \lambda_m)$, $\lambda = \sqrt{\lambda_1^2 + \dots + \lambda_m^2}$), i.e. $I(\varphi)$ is a homogeneous random field of “class $\mathfrak{S}_p$”(see $^{(1,2,7)}$). Consider the linear manifold M_F of functions $\psi(\mathbf{x})$ absolutely integrable in the Lebesgue sense and such that

$$\tilde{\psi}(\vec{\lambda}) = \int_{R^m} e^{i(\vec{\lambda},\mathbf{x})}\psi(\mathbf{x})\,d\mathbf{x} \in L_2(F).$$

The formula of the spectral representation

$$I(\varphi) = \int_{R^m} \tilde{\varphi}(\vec{\lambda})\, \Phi(d\vec{\lambda})$$

makes it possible to extend the field $I(\varphi)$ to a random linear functional on M_F, continuous in the mean square with respect to the norm

$$\|\psi\| = \int_{R^m} |\tilde{\psi}(\vec{\lambda})|^2\, F(d\vec{\lambda}), \qquad \psi \in M_F.$$

Denote further by $D^p_{L_1}$ the set of functions for which, for any integers $p_1, \dots, p_m \geq 0$, $p_1 + \dots + p_m \leq p$, there exist Lebesgue-absolutely integrable derivatives

$$D^{\mathbf{p}}\psi = \frac{\partial^{p_1+\dots+p_m}\psi}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}};$$

if

$$\int_{R^m} \frac{F(d\vec{\lambda})}{(1+\lambda^2)^p} < \infty,$$

then $D^p_{L_1} \subset M^1_F$.

Definition 1. Let $N = \{n\}$ be some directed set*, p a nonnegative integer; a generalized sequence of functions $\{\psi_n(\mathbf{x}),\ n \in N\}$ will be called p-ergodic if:

$$(E_1^p) \quad \text{All } \psi_n(\mathbf{x}) \in D_{L_1}.$$

$$(E_2^p) \quad \lim_{n \in N} \int_{R^m} \psi_n(\mathbf{x})\, d\mathbf{x} = 1.$$

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[*] A partially ordered set N is called directed if, for any two elements $n_1, n_2 \in N$, there exists an element n such that $n_1 \leq n,\ n_2 \leq n$.

(E_3^p) There exists a constant $C > 0$ such that: a) $\int_{R^m} |\psi_n(\mathbf{x})|\, d\mathbf{x} < C$ for all $n \in N$; b) $\int_{R^m} |D^p\psi_n(\mathbf{x})|\, d\mathbf{x} < C$ for all $n \in N$, if $\mathbf{p} = (p_1, \dots, p_m)$ and $p_1 + \dots + p_m \leq p$.

(E_4^p)

$$\lim_{n \in N} \int_{R^m} |\psi_n(\mathbf{x}+\mathbf{y}) - \psi_n(\mathbf{x})|\, d\mathbf{x} = 0$$

for every $\mathbf{y} \in R^m$.

Example. If $\varphi(\mathbf{x}) \in K$, $\int_{R^m} \varphi(\mathbf{x})\,d\mathbf{x} \neq 0$, and $\{\varphi_n(\mathbf{x}),\, n \in N\}$ is a 0-ergodic generalized sequence of functions on R^m, then the generalized sequence of functions

$$\psi_n^*(\mathbf{x}) = \frac{1}{\displaystyle\int_{R^m} \varphi\,d\mathbf{x}} \varphi^* \psi_n(\mathbf{x}) = \frac{1}{\displaystyle\int_{R^m} \varphi\,d\mathbf{x}} \int_{R^m} \varphi(\mathbf{x}-\mathbf{y})\psi_n(\mathbf{y})\,d\mathbf{y}$$

is p-ergodic for every p.

Examples of 0-ergodic generalized sequences of functions are given below (see Sec. 2, Examples 1 and 5).

Theorem 1. *Let $I(\varphi)$ be a homogeneous generalized random field of class $\mathfrak{S}_p$, and let $\{\psi_n(\mathbf{x}),\, n \in N\}$ be a p-ergodic generalized sequence of functions on R^m. Then the l.i.m. $I(\psi_n)$ exists and, with probability 1,*

$$\operatorname*{l.i.m}_{n \in N} I(\psi_n) = \Phi(\{0\}).$$

Let $K(\varphi)$ be a primitive of the generalized process $I(\varphi)$, $\varphi(x) \in K$, $x \in R^1$; consider the generalized processes

$$\int_{x-T}^{x+T} I(\varphi)\,dt = K(\varphi(x-T)) - K(\varphi(x+T)).$$

Corollary 1. For every stationary generalized random process and for any function $\varphi \in K$, there exists

$$\operatorname*{l.i.m}_{T \to +\infty} \frac{1}{2T} \int_{x-T}^{x+T} I(\varphi)\,dt = \Phi(\{0\}) \int_{-\infty}^{\infty} \varphi(t)\,dt;$$

in other words, the sequence of generalized random processes

$$\frac{1}{2T} \int_{x-T}^{x+T} I(\varphi)\,dt$$

as $T \to +\infty$ converges in mean square to the random constant $\Phi(\{0\})$.

An analogous proposition for generalized random processes whose realizations are generalized functions in the sense of Mikusiński was proved by Urbanik ([6],[7]).

Corollary 2. If $\xi(\mathbf{x})$, $\mathbf{x} \in R^m$, is a homogeneous continuous random field and $\{\psi_n(\mathbf{x}),\, n \in N\}$ is a 0-ergodic generalized sequence of functions on R^m, then there exists

$$\lim_{n \in N} \int_{R^m} \xi(\mathbf{x})\psi_n(\mathbf{x})\,d\mathbf{x} = \Phi(\{0\})$$

(the equality holds with probability 1).

2. The last proposition of Sec. 1 admits a generalization to a broad class of groups.

Definition 2. Let $G = \{g\}$ be a locally bicompact group, $\mu(\cdot)$ its left Haar measure. A generalized sequence of functions

$\{\psi_n(g),\, n \in N\}$ on the group G is called **left-ergodic** if:

(E_1) All $\psi_n(g) \in L_1(\mu)$.

$$(E_2) \quad \lim_{n\in N} \int_G \psi_n(g)\, \mu(dg) = 1.$$

$$(E_3) \quad \text{There exists a constant } C > 0 \text{ such that } \int_G |\psi_n(g)|\, \mu(dg) < C$$

for all $n \in N$.

$$(E_4) \quad \text{For every } f \in G \quad \lim_{n\in N} \int_G |\psi_n(fg) - \psi_n(g)|\, \mu(dg) = 0.$$

Example 1. Let $\{B_n,\, n \in N\}$ be a generalized sequence of measurable sets on G, and let $0 < \mu(B_n) < \infty$. The normalized characteristic functions of these sets

$$\left\{\frac{1}{\mu(B_n)}\chi_{B_n}(g),\, n \in N\right\}$$

form a left-ergodic generalized sequence of functions if and only if, for every $f \in G$, there exists

$$\lim_{n\in N} \frac{\mu(fB_n \Delta B_n)}{\mu(B_n)} = 0 \quad (A\Delta B = (A\cup B)\setminus(A\cap B)). \tag{1}$$

Definition 3. A generalized sequence of sets $\{B_n,\, n \in N\}$ possessing property (1) is called **left-ergodic**. This definition generalizes the definitions of ergodic sequences of sets proposed by Bokle (^{9}) and Calderón (^{10}) for unimodular groups.

Example 2. If $\{B_m^{(1)},\, m \in M\}$ and $\{B_n^{(2)},\, n \in N\}$ are left-ergodic generalized sequences of sets on locally bicompact groups G_1 and G_2, then the generalized sequence of sets

$$C_{m,n} = B_m^{(1)} \times B_n^{(2)}$$

(we put $(m_1, n_1) > (m_2, n_2)$ if $m_1 > m_2,\ n_1 > n_2$) is left-ergodic on $G_1 \times G_2$. Left-ergodic generalized sequences of sets are constructed analogously on semidirect products of locally bicompact groups*; note that, with the exception of very special cases, semidirect products of groups do not possess generalized sequences of sets that are ergodic in the sense of Bokle.

Example 3. If $\{B_n,\, n \in N\}$ is a left-ergodic generalized sequence of sets on the quotient group G/K of a (unimodular) locally bicompact group G by a bicompact normal subgroup K, then the generalized sequence $\{C_n,\, n \in N\}$ of full inverse images of the sets B_n under the natural homomorphism of G onto G/K is left-ergodic on G.

Example 4. On a bicompact group G, the sequence $\{G\}$, consisting of the single set G, is left-ergodic.

Example 5. Let R be the additive group of real numbers. Denote by $r(B)$ the upper bound of the radii of balls contained in the set $B \subset R^m$; as Day showed (^{11}), every generalized sequence of bounded convex sets $\{B_n,\, n \in N\}$ for which

$$\lim_{n\in N} r(B_n) = +\infty$$

is ergodic on R^m.

Example 6. Relying on Examples 2, 4, and 5, one can easily construct ergodic generalized sequences of sets on any commutative groups of bicompact origin (see (^{5})), since, up to algebraic and topological isomorphisms, every such group can be identified with a group of the form

$$K \times Z^p \times R^q,$$

where K is a bicompact group and Z is the additive group of integers.

* For the definition of a semidirect product of topological groups, see, for example, (^{4}).

Example 7. Let G be an arbitrary commutative locally bicompact group. Consider the set $\{F_\alpha, \alpha \in A\}$ of all open subgroups of the group G having bicompact origin. Let $\{\psi_{k,\alpha}, k = 1, 2, ...\}$ be an ergodic sequence of functions on F_α such that all $\psi_{k,\alpha} \geq 0$,

$$\psi_{k,\alpha}(g) = 0, \quad g \notin F_\alpha, \quad \text{and} \quad \int_G \psi_{k,\alpha}(g)\, \mu(dg) = 1$$

(see examples 1 and 6); then the generalized sequence of functions

$$\psi_{k,\alpha_1,\dots,\alpha_m}(g) = \psi_{k,\alpha_1} * \cdots * \psi_{k,\alpha_m}(g)$$

(we set $(k', \alpha_{i_1}, \dots, \alpha_{i_m}) > (k'', \alpha_{j_1}, \dots, \alpha_{j_{m'}})$ if $k' > k''$ and $\{\alpha_{i_1}, \dots, \alpha_{i_m}\} \supset \{\alpha_{j_1}, \dots, \alpha_{j_{m'}}\}$) is ergodic on G.

Example 8. Free products of cyclic groups do not possess left-ergodic generalized sequences of sets (the only exceptions are cyclic groups and free products of two cyclic groups of order 2); the same may be said of the discrete group of rotations of three-dimensional space about a fixed point (cf. with (^{12})).

Theorem 2. Let $\{\psi_n(g), n \in N\}$ be a left-ergodic generalized sequence of functions on a locally bicompact group G. Then for every measurable random field $\xi(g)$* homogeneous with respect to right shifts: 1) there exists

$$\mathrm{l.\,i.\,m.}_{n\in N}\int_G \xi(g)\psi_n(g)\,\mu(dg) = \bar{\xi};$$

2) $D\bar{\xi} < +\infty$; 3) the random variable $\bar{\xi}$, up to equivalence, is uniquely determined by the field $\xi(g)$—independently of the choice of the left-ergodic generalized sequence of functions; 4) the random variable $\bar{\xi}$ is invariant with respect to left and right shifts.

This theorem generalizes the ergodic theorems of Wiener (^{18}) and Calderon (^{10}) on convergence in quadratic mean.

The method used in the proof of theorem 2 also makes it possible to obtain the following generalization of results of Lyubarskii (^{3}) and Struble (^{16}).

Theorem 3. Let $\{\psi_n(g), n \in N\}$ be a left-ergodic generalized sequence of functions on a locally bicompact group G. Then for every measurable almost-periodic (positive definite) function $p(g)$, $g \in G$, the Neumann (Godement) mean $M(p)$ (see (^{13,14})) can be defined by the formula

$$M(p) = \lim_{n\in N}\int_G p(g)\psi_n(g)\,\mu(dg).$$

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* A random field $\xi(g)$ on the group G is called homogeneous with respect to right shifts if $M|\xi(g)|^2 < +\infty$, $M\xi(g) = \text{const}$, and

$$M\xi(g_1 g)\overline{\xi(g_2 g)} = M\xi(g_1)\overline{\xi(g_2)}$$

for any $g_1, g_2, g \in G$ (see (8)).

Note: Figure translations are in progress. See original paper for figures.

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