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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

Mathematics

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A Vertex Function of Directed Graphs

(Presented by Academician S. L. Sobolev on 2 IV 1962)

In this paper one of the problems posed by A. A. Zykov ([2]) is solved. Graphs with directed arcs (see ([3]), Ch. I), without loops, are considered.

Let $p, s, t, q, r, 1$ be elements of some ring K. Consider a function $\Phi(L)$ on graphs with ordered vertices, taking values in the ring K and defined by the relations

$$\begin{gathered}\Phi(L) = p\Phi(L_p) + s\Phi(L_s) + \\ + t\Phi(L_t) + q\Phi(L_q) + r\Phi(L_r) + 1; \\ \Phi(E_0) = 0,\end{gathered} \tag{1}$$

where E_0 is the null graph; L_p, L_s, L_t, L_q, L_r are subgraphs of the graph L, whose definition is connected with deletion of the first vertex a and is clear from Fig. 1; the ordering of the vertices in these subgraphs is induced by the ordering in L.

Fig. 1

We shall determine the conditions for the independence of $\Phi(L)$ from the method of ordering the vertices of L (the uniqueness conditions, see ([1])).

It is easy to verify on the corresponding graphs of Fig. 2 that, for the uniqueness of $\Phi(L)$, the following conditions must hold:

$$p = s, \qquad (3_1)$$
$$(1-r)p = (1-r)q = (1-r)t, \qquad (3_2)$$
$$p(p+r-1) = 0, \qquad (3_3)$$
$$p(q+r-1) = 0, \qquad (3_4)$$
$$p(t+r-1) = 0, \qquad (3_5)$$
$$q(p+t-1) = 0, \qquad (3_6)$$
$$q(t+r-1) = 0, \qquad (3_7)$$
$$t(p+r-1) = 0, \qquad (3_8)$$
$$t(q+r-1) = 0. \qquad (3_9)$$

Fig. 2

Fig. 2

Figure 2: Fig. 2

It is not difficult, by the method of mathematical induction, to prove that the general expression for $\Phi(L)$, found from (1) and (2), has the form

$$\begin{aligned}\Phi(L) = \varphi_p(p,q,r,t)(p+r-1) + \varphi_q(p,q,r,t)(q+r-1) + \\ + \varphi_t(p,q,r,t)(t+r-1) + d_1(L),\end{aligned} \tag{4}$$

where $\varphi_p, \varphi_q, \varphi_t$ are polynomials in p, q, r, t with nonnegative integer coefficients, and $d_1(L)$ is the number of vertices in L. Moreover, it turns out that $\varphi_p(0) = d_1(L_p) + d_1(L_s)$, $\varphi_q(0) = d_1(L_q)$, $\varphi_t(0) = d_1(L_t)$.

After this it is easily verified that the conditions (3_1)—(3_9) are sufficient for $\Phi(L)$ not to depend on the order of numbering of the vertices of L. For this, in an arbitrary graph L one must distinguish any two vertices, and divide all the remaining vertices into 10 groups depending on the character of adjacency with each of the distinguished vertices; then apply (1) twice, the first time beginning with one vertex and the second time with the other, and show that the difference of the results obtained is equal to zero by virtue of the relations (3) and the representation (4).

To find out what properties of a graph are described by the vertex function, it is necessary to transform expression (4) for $\Phi(L)$. On the basis of (3_1)—(3_9) it is not difficult to reduce

$$\varphi_p(p,q,r,t)(p+r-1) \quad \text{to the form} \quad [f_p(r) + d_1(L_p) + d_1(L_s)](p+r-1),$$

$$\varphi_q(p,q,r,t)(q+r-1) \quad \text{to the form} \quad [F_q(q) + f_q(r) + d_1(L_q)](q+r-1),$$

$$\varphi_t(p,q,r,t)(t+r-1) \quad \text{to the form} \quad [F_t(t) + f_t(r) + d_1(L_t)](t+r-1),$$

where $f(0) = F_q(0) = F_t(0) = 0$.

After this $\Phi(L)$ is rewritten in the form

$$\Phi(L) = [f_p(r) + d_1(L_p) + d_1(L_s)](p+r-1) + [F_q(q) + f_q(r) +$$

$$+ d_1(L_q)](q+r-1) + [F_t(t) + f_t(r) + d_1(L_t)](t+r-1) + d_1(L). \tag{5}$$

Let us introduce the concepts of 0-complete and 2-complete dimensional polynomials. By a 0-complete dimensional polynomial we shall mean

$$\overline{D}(q) = 1 + \overline{d}_1 q + \overline{d}_2 q^2 + \cdots,$$

where $\overline{d}_i$ is the number of such i-vertex subgraphs of the graph L in which no two vertices are joined by an arc, in other words, the number of internally stable i-vertex subsets of L (see ([3]), Chap. 4).

It is verified directly that $\overline{D}(q)$ satisfies the conditions

$$\overline{D}(L) = q\overline{D}(L_q) + \overline{D}(L_r),$$

$$\overline{D}(E_0) = 1. \tag{6}$$

By a 2-complete dimensional polynomial we shall mean the polynomial $D(t) = 1 + d_1 t + d_2 t^2 + \cdots$, where d_i is the number of such i-vertex subgraphs of the graph L any two vertices of which are joined by arcs of both directions; in other words, d_i is the number of i-vertex internally stable subsets of the graph $\overline{L} = (X, \overline{\Gamma})$, complementary to the graph $L(X, \Gamma)$ and defined as follows:

$$y \in \overline{\Gamma} x \Longleftrightarrow y \notin \Gamma x \quad \text{and} \quad y \neq x.$$

Analogously to the preceding case it is verified that $D(t)$ satisfies the conditions

$$D(L) = tD(L_t) + D(L_r),$$

$$D(E_0) = 1. \tag{7}$$

After this one can determine the meaning of the polynomials $F_q(q)$ and $F_t(t)$ in expression (5).

If on the generators of the ring K the conditions $r = 1,\ p = t = 0$ are imposed, then all the conditions (3_1)—(3_9) will be fulfilled, and the relations (1), (2) take the form

$$\Phi(L) = q\Phi(L_q) + \Phi(L_r) + 1,$$

$$\Phi(E_0) = 0.$$

It is verified directly that this relation is satisfied by the function

$$\Phi(L) = \frac{\overline{D}(q) - 1}{q} = \overline{d}_1 + \overline{d}_2 q + \overline{d}_3 q^2 + \cdots;$$

on the other hand, from (5), putting $r = 1,\ p = t = 0$, we obtain

$$\Phi(L) = [F_q(q) + f_q(1) + d_1(L_q)]q + d_1(L).$$

Hence

$$F_q(q) = \overline{d}_2 - d_1(L_q) - f_q(1) + \overline{d}_3 q + \overline{d}_4 q^2 + \cdots,$$

and since $F_q(0) = 0$, it follows that

$$f_q(1) = \overline{d}_2(L) - d_1(L_q), \tag{8}$$

and, consequently,

$$F_q(q) = \overline{d}_3 q + \overline{d}_4 q^2 + \cdots. \tag{9}$$

Similarly, putting $p = q = 0,\ r = 1$ and using (7), we obtain:

$$F_t(t) = \overline{d}_3 t + \overline{d}_4 t^2 + \cdots, \tag{10}$$

$$f_t(1) = d_2(L) - d_1(L_t). \tag{11}$$

By simple algebraic transformations we reduce (5) to the form

$$\begin{aligned}\Phi(L) = {} & F_q(q)q + F_t(t) + [f_p(r)p + f_q(r)q + f_t(r) + d_1(L_p)p \\ & + d_1(L_s)p + d_1(L_q)q + d_1(L_t)t] + [rf_p(r) - f_p(r) + rf_q(r) \\ & - f_q(r) + rf_t(r) - f_t(r) + (d_1(L) - 1)r] + 1.\end{aligned} \tag{12}$$

It remains to clarify the meaning of the expressions in the first and second square brackets of (12). Denote $d_1(L)$ by n. Let

$$f_p(r) = a_1 r + a_2 r^2 + \cdots + a_{n-1} r^{n-1},$$

$$f_q(r) = b_1 r + b_2 r^2 + \cdots + b_{n-1} r^{n-1},$$

$$f_t(r) = c_1 r + c_2 r^2 + \cdots + c_{n-1} r^{n-1}.$$

It is not difficult to show that

$$[rf_p(r) - f_p(r) + rf_q(r) - f_q(r) + rf_t(r) - f_t(r) + (n-1)r] \\ = r + r^2 + \cdots + r^{n-1}. \tag{13}$$

Hence, equating coefficients of like powers of r, we shall have

$$a_{n-k} + b_{n-k} + c_{n-k} = k - 1.$$

Therefore,

$$f_p(r) + f_q(r) + f_t(r) = (n-2)r + (n-3)r^2 + \cdots + r^{n-2}.$$

Replacing everywhere rq by $q - p + rp$, and rt by $t - p + rp$, we obtain

$$f_p(r)p + f_q(r)q + f_t(r)t =$$

$$\begin{aligned} &= [f_p(r) + f_q(r) + f_t(r)]p + f'_q(r)q - f'_q(r)p + f'_t(r)t - f'_t(r)p \\ &= [(n-2)r + (n-3)r^2 + \cdots + r^{n-2}]p \\ &\quad + [b_1 + b_2 + \cdots + b_{n-1}](q-p) + [c_1 + c_2 + \cdots + c_{n-1}](t-p) \\ &= [(n-2)r + \cdots + r^{n-2}]p + [\bar{d}_2(L) - d_1(L_p)](q-p) \\ &\quad + [d_2(L) - d_1(L_t)](t-p), \end{aligned}$$

where $f'(r) = f(r)/r$.

After this the first square bracket in (12) can be rewritten in the form:

$$\bar{d}_2(L)q + d_2(L)t + [n - 1 + \bar{d}_2(L) + d_2(L) + (n-2)r + \\ + (n-3)r^2 + \cdots + r^{n-2}]p.$$

Taking this into account and using (9), (10), and (13), we reduce the expression for $\Phi(L)$ to the form

$$\Phi(L) = \bar{d}_2 q + \bar{d}_3 q^2 + \cdots + d_2 t + d_3 t^2 + \cdots \\ + [n - 1 - \bar{d}_2 - d_2 + (n-2)r + (n-3)r^2 + \cdots + r^{n-2}]p + \\ + r + r^2 + \cdots + r^{n-1} + 1. \tag{14}$$

Thus, the information about the graph carried by the vertex function is completely characterized by the numbers $\bar{d}_i(L)$ of internally stable i-vertex sets of the given graph and by the numbers $d_i(L)$ of internally stable i-vertex sets of the graph supplementary to L $(i = 1, 2, 3, \dots)$.

In conclusion, we note that consideration of a function $\Phi(L)$ satisfying, instead of conditions (1) and (2), the conditions

$$\Phi(L) = t\Phi(L_t) + q\Phi(L_q) + \Phi(L_r) + 1,$$
$$\Phi(E_0) = 0,$$

does not lead to any loss of information about the graph L. Indeed, when the additional relations $r = 1,\ p = 0$ are imposed on the generators of the ring K, the uniqueness conditions (3) take the form

$$qt = tq, \tag{15}$$

and expression (14) passes into

$$\Phi!(L) = \bar{d}_2 q + \bar{d}_3 q^2 + \cdots + d_2 t + d_3 t^2 + \cdots + n,$$

while the latter admits no transformations whatever in which conditions (15) would be essentially used.

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Note: Figure translations are in progress. See original paper for figures.