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## Abstract

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## ON THE STRUCTURE OF RELATIVISTIC NONLINEAR WAVES IN PLASMA

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**1.** When a relativistic beam of charged particles passes through a plasma, waves are excited in it which propagate with a velocity close to the speed of light. The linear approximation for considering waves in the system can serve only to estimate the time required for the establishment of nonlinear motion from the decrement of the buildup of oscillations ([1]). Nonrelativistic nonlinear waves in plasma have been considered in ([2,3]). Here we wish to draw attention to a number of interesting properties of relativistic nonlinear waves, and also, using the example of two mutually penetrating identical plasmas, to illustrate the solution of the nonlinear problem of the loss of energy by a beam in a plasma.

In writing the initial system of equations we assume: 1) for describing the plasma and the beam, the hydrodynamic approximation at temperatures equal to zero; 2) the ions compensating, at the initial moment of time, the volume charges of the plasma and beam are immobile; 3) the motion is one-dimensional (dependence on the time $x_0$ and the coordinate $x$). To describe the motion of electrons we use the equations of relativistic gas dynamics in self-consistent fields ([4]):

$$\chi = \frac{\partial u_i}{\partial x_0} + \frac{\partial}{\partial x}\sqrt{1+u_i^2}; \qquad \frac{\partial}{\partial x} n_i u_i + \frac{\partial}{\partial x_0} n_i \sqrt{1+u_i^2} = 0; \tag{1}$$

$$\frac{\partial \chi}{\partial x_0} = r_0 \sum_i \left(n_i u_i - n_i^{(0)} u_i^{(0)}\right);$$

$$\frac{\partial \chi}{\partial x} = -r_0 \sum_i \left(n_i \sqrt{1+u_i^2} - n_i^{(0)} \sqrt{1+u_i^{(0)2}}\right), \tag{2}$$

where $\chi = \dfrac{e}{m} E_x$; $E_x$ is the electric-field strength; $u_i$ is the component of the 4-velocity along $x$; $n_i$ is the proper density, $i = 1, 2$; $r_0 = 4\pi \dfrac{e^2}{mc^2}$; the index 0 denotes the initial values of the quantity (the compensating charge at $x_0 = 0$); $\gamma_i = \sqrt{1+u_i^2}$.

For the established motion there exists a reference frame in which all quantities are independent of time. In this frame

$$\gamma_i - \varphi = \text{const} = \gamma_{i0}; \qquad n_i u_i = \text{const} = n_{i0} u_{i0};$$

$$\sum_i \left( n_{i0} u_{i0} - n_i^{(0)} u_i^{(0)} \right) = 0, \tag{3}$$

where $\varphi$ is the potential of the electric field: $\chi = -\partial\varphi/\partial x$. Substitution of (3) into (2) gives a nonlinear equation for the potential, whose first integral has the form

$$\frac{1}{2}\left(\frac{d\gamma_1}{dx}\right)^2 = r_0 \sum_i \left( n_{i0} u_{i0} \sqrt{\gamma_i^2 - 1}\, \operatorname{sign} u_i - n_i^{(0)} \gamma_i^{(0)} \gamma_i \right) + \text{const}, \tag{4}$$

where $\gamma_i = \gamma_1 + \gamma_{i0} - \gamma_{10}$ and $\varphi = \gamma_1 - \gamma_{10}$. Integration of (4) reduces to quadratures.

2. Let us consider nonlinear relativistic waves in a plasma without a beam ($i = 1$). From (3), $n_{10} u_{10} = n_1^{(0)} u_1^{(0)}$, and with the corresponding choice of the origin for $\varphi$ one may set $u_{10} = u_1^{(0)}$, whence $n_{10} = n_1^{(0)}$,

$$\frac{d\gamma_1}{d\xi} = \pm\sqrt{\lambda + u_{10}\sqrt{\gamma_1^2 - 1} - \gamma_{10}\gamma_1}; \qquad \xi = \sqrt{2 n_{10} r_0}\, x. \tag{5}$$

The constant $\lambda$ characterizes the intensity (amplitude) of the wave, since $\gamma_{1,\max} = \gamma_{10}\lambda + u_{10}\sqrt{\lambda^2 - 1}$; $\gamma_{1,\min} = \gamma_{10}\lambda - u_{10}\sqrt{\lambda^2 - 1}$, with $\lambda_{\max} = \gamma_{10}$, and the maximum potential difference in the wave $\Delta\varphi = \gamma_{1,\max} - \gamma_{1,\min}$ is $\Delta\varphi_{\max} = 2u_{10}$. The constant $\lambda$ is related to the quantity $\beta$ of Ref. ([2]) by the relation $\lambda^2 = 1 + \beta^2 u_{10}^2$.

The independence of the frequency of nonlinear oscillations from the amplitude, obtained in ([2]), is preserved only in the nonrelativistic approximation. In the general case, for the period $\xi_0$ we obtain

$$\xi_0 = 2\int_{\gamma_{1,\min}}^{\gamma_{1,\max}} d\gamma_1 \left(\frac{d\gamma_1}{d\xi}\right)^{-1} = \frac{2\sqrt{2}}{\sqrt[4]{1 - k^2}}\, E(k)\, \frac{\alpha^2 - 1}{\alpha}, \tag{6}$$

where $\alpha = u_{10} + \sqrt{u_{10}^2 + 1}$; $k^2 = 2\sqrt{\lambda^2 - 1}\,(\lambda + \sqrt{\lambda^2 - 1})^{-1}$; $E(k)$ is the complete elliptic integral; $k^2 \ll 1$ corresponds to the nonrelativistic approximation of Ref. ([2]), whereas $k \to 1$ for relativistic waves of high intensity. In the latter case the period $\xi_0$ grows with the amplitude of the wave as $\xi_0 \simeq 8u_{10}\lambda^{1/2}$. It should be emphasized that, for nonrelativistic waves, the dependence of the frequency on the amplitude characteristic of nonlinear oscillations is a relativistic effect. The structure of the wave ($-\xi_0/2 < \xi < \xi_0/2$) has the form

$$\pm\frac{\xi}{\sqrt{2}}=\frac{1}{\sqrt[4]{1-k^2}}\,\frac{\alpha^2-1}{\alpha}\,E(k,\theta)+\frac{k^2}{2\sqrt[4]{1-k^2}}\,\frac{1}{\alpha}\,\frac{\sin 2\theta}{\sqrt{1-k^2\sin^2\theta}};$$

$$\theta=\frac{1}{2}\arccos\frac{\gamma_1+\sqrt{\gamma_1^2-1}-\lambda\alpha}{\sqrt{\lambda^2-1}\,\alpha}. \tag{7}$$

In the nonrelativistic limit ($k^2 \ll 1$), from (7) we obtain the result of Ref. [2] ($E(k,\theta)\simeq\theta,\ \lambda-1\simeq\frac{1}{2}\beta^2u_{10}^2\ll 1,\ \gamma^2-1=u_{10}^2\psi\ll 1$), whereas for ultrarelativistic strong waves ($k\to 1$), almost throughout the region $\xi^2<\frac{1}{4}\xi_0^2$,

$$\frac{\gamma_1+\sqrt{\gamma_1^2-1}}{4\lambda u_{10}}\simeq 1-\frac{4\xi^2}{\xi_0^2};\qquad 1-\frac{4\xi^2}{\xi_0^2}\gg\frac{1}{2\lambda^2};\qquad \lambda\gg 1. \tag{8}$$

The energy density of the electric field of the wave, averaged over a period, is equal to

$$\overline{W}_E=\frac{2}{\xi_0}\int_0^{\xi_0/2}\frac{E^2}{8\pi}\,d\xi=\frac{1}{6}mn_{10}\frac{k^2}{\sqrt{1-k^2}}\left[\frac{2}{k^2}-1+2\left(1-\frac{1}{k^2}\right)\frac{K(k)}{E(k)}\right], \tag{9}$$

where $K(k)$ and $E(k)$ are complete elliptic integrals. In the nonrelativistic limit ($k \ll 1$), $\overline{W}_E\simeq\frac{1}{4}mn_{10}\beta^2u_{10}^2$, which coincides with the result of the linear approximation. As $k\to 1$, $\overline{W}_E\simeq\frac{1}{3}n_{10}m\lambda$.

The plasma-particle kinetic-energy density averaged over the period is equal to

$$-\overline{T}_{44}=W(1);\qquad W(\varepsilon)=\frac{mn_{10}u_{10}}{2\alpha(\alpha^2-1)\sqrt{1-k^2}}\times$$

$$\times\left\{\frac{2}{3}(2-k^2)(1+\alpha^4)+(1-k^2)\left(2\alpha^2\varepsilon-\frac{1+\alpha^4}{3}\right)\frac{K(k)}{E(k)}\right\}. \tag{10}$$

For $k\to 1$ we obtain $-\overline{T}_{44}\simeq {}^4/_3\,mn_{10}\gamma_{10}^2\lambda$. Similarly, for the mean momentum density and the mean momentum flux $\overline{T}_{11}=W(-1)$

$$\overline{T}_{14}=i\,\frac{mn_{10}u_{10}(\alpha^2+1)}{6\alpha\sqrt{1-k^2}}\left[2(2-k^2)-(1-k^2)\frac{K(k)}{E(k)}\right]. \tag{11}$$

Hence one finds the velocity $\beta_0$ of the reference frame in which the mean momentum density is zero:

$$\beta_0=\frac{1}{2}\frac{\overline{T}_{44}-\overline{T}_{11}}{i\overline{T}_{14}}-\sqrt{-1-\frac{(\overline{T}_{44}-\overline{T}_{11})^2}{4\overline{T}_{14}^2}}=\frac{u_{10}}{\sqrt{1+u_{10}^2}}$$

$$\text{or} \qquad u_{10} = \frac{\beta_0}{\sqrt{1-\beta_0^2}}. \tag{12}$$

Thus the four-dimensional component of the wave velocity coincides with $u_{10}$. The density averaged over the period is

$$\overline{n_1\gamma_1} = n_{10}u_{10}\frac{\alpha^2+1}{\alpha^2-1}; \qquad \overline{n_1\gamma_1} = n_{10}\gamma_{10} \quad \text{for} \quad \alpha = u_{10} + \sqrt{u_{10}^2+1}. \tag{13}$$

3. Let us consider nonlinear waves in a system of two interpenetrating identical plasmas: $n_1^{(0)} = n_2^{(0)}$, $u_1^{(0)} = -u_2^{(0)}$. With the corresponding choice of the origin for $\varphi$, one may set $u_{10} = u_1^{(0)}$. From the symmetry of the problem $u_1 = -u_2$ and, consequently, $u_{20} = -u_1^{(0)}$. Finally, from (3) we obtain $n_{10} = n_{20}$. The calculations coincide with those carried out above (in equation (5) the factor $\nu = n_1^{(0)}/n_{10}$ enters before $\gamma_{10}$), with $\lambda_{\max} = \nu\gamma_{10}$; $\Delta\varphi_{\max} = 2u_{10}^2(\nu^2 + (\nu^2-1)u_{10}^2)^{-1}$. The mean values are described by doubled expressions (9), (10), (11), with $\alpha = \nu\gamma_{10} + u_{10}$ and $k^2 = 2\sqrt{\lambda^2 - (\nu^2\gamma_{10}^2 - u_{10}^2)}/[\lambda + \sqrt{\lambda^2 - (\nu^2\gamma_{10}^2 - u_{10}^2)}]$; in the expression for the period (6) one must replace $\xi_0$ by $\xi_0\sqrt{\nu^2\gamma_{10}^2 - u_{10}^2}$ and $r_0$ by $2r_0$ (the reduced mass enters).

Let the initial densities and velocities, along with constant components, contain relatively small fluctuating quantities. In considering the nonlinear motion one may neglect the fluctuations if the changes of the quantities in the wave considerably exceed the fluctuating ones. The presence of fluctuations is essential in the initial stage.

Let us average equation (1) over an interval $L$ considerably exceeding all characteristic distances of the problem. Then, assuming boundedness and single-valuedness of the quantities, we obtain conservation in time of $\overline{n_i\gamma_i}$, $\overline{W_E} - \overline{T}_{44}$, $\overline{T}_{14}$. Neglecting quantities quadratic in the fluctuations, one may take the initial means to be equal to the constant components. The means of the final nonlinear motion obviously coincide with the means over the period. In using the conservation laws it is assumed that the initial perturbation tends to a nonlinear final one. Such an assumption was made in ([2]). A situation in which this is not true is conceivable (for example, if multi-velocity flows develop).

Using $n_1\gamma_1 = \text{const}$ and (13) gives an equation for $\nu$, having the unique solution $\nu = 1$. Conservation of $\overline{T}_{14}$ is trivial ($0 = 0$). Conservation of $\overline{W_E} - \overline{T}_{44}$ gives a transcendental equation for determining $\lambda$. In the ultrarelativistic limit $\alpha \simeq 2u_{10}^2 \gg 1$, an approximate solution of this equation can be found by expanding the elliptic integrals in $k^2$. We have

$k^2 \approx \dfrac{8}{3u_{10}^2}$ and $\Delta\varphi \approx \dfrac{16}{3}\dfrac{1}{u_{10}}$. Thus, a relatively small fraction of the energy of the colliding beams can be transferred into the oscillations. The decrease in the momentum of each component is relatively small,

$$\Delta\overline{T}_{14} \approx i\,\frac{4}{9}\,\frac{mn_{10}}{u_{10}^2}.$$

In a similar way, the problem of energy losses in the collision of plasmas of different densities can be solved.

Finally, the time required for the establishment of nonlinear motion can be estimated from the decrement of the growth of the oscillations from the dispersion equation (see ([1], [5])) of the linearized system (1), (2):

$$1 = \frac{r_0 n_{10}}{\gamma_{10}^2 \left(\omega - k\dfrac{u_{10}}{\gamma_{10}}\right)^2} + \frac{r_{20} n_{20}}{\gamma_{20}^2 \left(\omega + k\dfrac{u_{20}}{\gamma_{20}}\right)^2}$$

(initial fluctuations in $x$).

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*Note: Figure translations are in progress. See original paper for figures.*