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FINITE AUTOMATA AND THE LOGIC OF MONADIC PREDICATES

1961

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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

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FINITE AUTOMATA AND THE LOGIC OF MONADIC PREDICATES

(Presented by Academician P. S. Novikov on 16 V 1961)

1°. In papers ([1–4]) the connection between finite automata and formal languages based on the logic of monadic predicates was investigated. These works contain the following restrictions on the logical means admissible in the languages employed:

1. The language developed in ([1]) presupposes the controllability of object quantifiers.
2. In Church' s restricted recursive arithmetic ([3]) there are no predicate quantifiers.
3. In Büchi' s weak second-order arithmetic ([4]) there are no restrictions either on object or on predicate quantifiers; however, the predicates themselves are interpreted as special predicates capable of selecting only finite (and not arbitrary) sets of natural numbers.

In the present paper theorems are established for the language (see ([2])), in which all the indicated restrictions are removed. The main attention is devoted to the following problems:

I. To determine what sets and operators can be described in the language (Theorems 2 and 3, and also item 9).

II. To obtain criteria for the existence of finite operators (in particular, finite automata) satisfying a condition expressed in a given formula of (Theorems 4 and 4′).

If such exist:

III. To consider procedures for passing from a formula to the corresponding canonical equations (Theorem 5).

In solving these problems an important role is played by the algorithm for reduction to normal form (item 3°) and the notion of a homogeneous set (item 5°). Convenient, although, of course, not necessary, is the use of a geometric interpretation close to that already applied in ([5]) for the investigation of partially recursive operators by means of Baire space.

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2°. The terminology and notation are basically the same as in ([1,2]). We note the following difference in terminology from ([1]): the t-formulas of ([1]) will here be called lower t-formulas; we shall also consider upper t-formulas, which differ from the lower ones only in that, in the definition of a t-controlled quantifier, it is necessary everywhere to replace "by" with "on." An example of an upper t-formula is:

$$(E\tau)_{\tau\geq t}[X(\tau) \mathbin{\&} (E\Gamma)_{\tau\geq\rho\geq t}\ (\rho)\ \Gamma(\rho)].$$

Until now it has remained unknown whether there exists an -algorithm, i.e., an algorithm for recognizing the truth of closed formulas of (Tarski's problem). In the present paper we shall use the term "conditional -algorithm" to denote any such procedure whose effectiveness would be ensured if an -algorithm existed.

A recursive operator transforming a system of predicates $\{X_i(\tau)\}$ into a system of predicates $\{Z_j(\tau)\}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ will be called finite if it can be described by canonical equations.

(cf. with system (2) from (1)):

$$\begin{gathered} Z_j(t) = \\ = \Phi_j\,[X_1(t), \dots, X_1(t+r), \dots, X_m(t), \dots, X_m(t+r), \Gamma_1(t), \dots, \Gamma_k(t)], \\ \Gamma_\nu(t+1) = \\ = \Psi_\nu\,[X_1(t), \dots, X_1(t+r), \dots, X_m(t), \dots, X_m(t+r), \Gamma_1(t), \dots, \Gamma_k(t)], \\ j = 1, 2, \dots, n, \qquad \nu = 1, 2, \dots, k \end{gathered}$$

under the initial conditions $\Gamma_\nu(1) = \sigma_\nu$ (initial constants). The number k characterizes the size of the operator' s memory. In contrast to an L-operator (an operator realized in a finite automaton), a finite operator is, generally speaking, also characterized by anticipation (over a time interval of length r).

3°. Normal form. In our constructions the following fact, already partly used in ([1,2]), plays an important role.

Theorem 1. There exists an algorithm that transforms any formula $\mathfrak{A}[X_1, X_2, \dots, X_m]$ of I into an equivalent formula of the special form

$$\Phi[\Gamma_1, \dots, \Gamma_a, \tilde{\Gamma}_1, \dots, \tilde{\Gamma}_b],$$

where Φ is a function of the algebra of logic, Γ_i are lower t-formulas, and $\tilde{\Gamma}_j$ are upper t-formulas.

4°. Geometric language. To a one-place predicate X, successively taking the values $X(1), X(2), \dots, X(t), \dots$, there corresponds one-to-one a point of the Cantor set P, having the ternary expansion $0, \xi_1\xi_2 \dots \xi_t \dots$, where $\xi_t = 2 \cdot X(t)$. In the same sense, ordered systems of predicates will be given by Cartesian

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products $P^x \times P^y$ ... (denoted by P^{xy}, etc.). An interval $P_{\sigma_1 \ldots \sigma_\nu}$ of rank ν is the set of all points of P for which $X(1) = \sigma_1$, $X(2) = \sigma_2, \ldots, X(\nu) = \sigma_\nu$; a "multidimensional" interval of rank ν is a Cartesian product of one-dimensional intervals of rank ν.

Let $\mathfrak{M} \subseteq P$; by $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu}$ we denote the set (possibly empty) of all such points X that $0, \sigma_1, \ldots, \sigma_\nu, \xi_{\nu+1}, \xi_{\nu+2}, \ldots \in \mathfrak{M} \cap P_{\sigma_1 \ldots \sigma_\nu}$ with $\xi_{\nu+t} = 2 \cdot X(t)$ (and analogously for multidimensional intervals).

In $P^x, P^{xy}, \ldots$ we shall consider the ordinary topology. For simplicity of exposition, wherever possible, we restrict ourselves to considering sets from P^x or P^{xy}.

5°. Homogeneous sets and transition function.

Definition. A set $\mathfrak{M}$ is homogeneous of degree k if among all its possible $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu}$ there exist no more than k pairwise distinct sets. Any system $\{\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_k\}$ such that: 1) $\mathfrak{M}_i$ is some $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu}$, and 2) every $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu}$, as well as $\mathfrak{M}$ itself, coincides with one of the $\mathfrak{M}_i$, forms a basis for $\mathfrak{M}$.

If $\mathfrak{M}_i$ is the set $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu}$, and $\mathfrak{M}_j$ is $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu \sigma}$, then we shall say that σ carries $\mathfrak{M}_i$ into $\mathfrak{M}_j$; we shall call a transition function (relative to the given basis) any function $\psi[i, \sigma]$ that assigns, for a given $i = 1, 2, \ldots, k$ and $\sigma = 0, 1$, the number of some basis set into which σ carries $\mathfrak{M}_i$.

6°. We shall say of the sets $\widehat{X}\,\mathfrak{A}(X)$, $\widehat{XY}\,\mathfrak{A}(X, Y), \ldots$ that they are I-definable (by the formulas $\mathfrak{A}(X)$, $\mathfrak{A}(X, Y), \ldots$).

Theorem 2. Every I-definable set $\mathfrak{M}$ is homogeneous; there exists an algorithm which, from the formula $\mathfrak{A}$, delivers the transition function for the set defined by this formula.

There exist homogeneous sets that are not I-definable. However, the following holds

Theorem 3. Every closed homogeneous set is I-definable by a formula of the form

$$\text{[predicate existential quantifiers] } (t) \text{ [quantifier-free part].} \qquad (*)$$

It follows from this theorem that the class of I-definable sets is the smallest class of sets that contains all closed homogeneous sets and is closed under the operations of complementation, finite set-theoretic summation, and also projection into a subspace. It remains unclear which Borel and projective classes are represented by I-definable sets.

The proof of Theorem 2 is based on describing a procedure that assigns to every homogeneous set an L-operator which a-enumerates (see ([2])) all tuples $\sigma_1, \ldots, \sigma_\nu$ such that $\mathfrak{M}_{\sigma_1 \ldots \sigma_\nu}$ is nonempty. If $\mathfrak{M}$ is I-definable by a formula $\mathfrak{A}(X)$, then

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this procedure is a conditional I-algorithm transforming the formula $\mathfrak{A}(X)$ into a formula $\mathfrak{A}^*(X)$ of type $(*)$, which specifies the closure $\hat{X}\mathfrak{A}(X)$.

7°. **Theorem 4.** If $\mathfrak{M} = \hat{X}\hat{Y}\mathfrak{A}(X, Y)$ is the graph of an everywhere-defined continuous function $Y = T[X]$ (or, what is the same thing, if $\mathfrak{M}$ is closed and uniform over all P^*), then T is a finite operator; moreover, the weight of the operator does not exceed the degree of homogeneity of $\mathfrak{M}$.

In particular, every I-definable recursive operator is a finite operator, and every I-definable operator without anticipation is realizable in a finite automaton.

Following Church ([3]), we shall say that the operator $Y = T[X]$ is a solution of $\mathfrak{A}[X, Y]$ (or satisfies $\mathfrak{A}[X, Y]$) if, for every X, $\mathfrak{A}[X, T[X]]$ is true. A strengthening of Theorem 4 is:

Theorem $4'$**.** Let $\hat{X}\hat{Y}\mathfrak{A}(X, Y)$ be closed (of degree of homogeneity k). If $\mathfrak{A}(X, Y)$ has a continuous solution, then it also has a solution that is a finite operator (of weight $\leq k$); if among its solutions there is an operator without anticipation, then a solution will also be some L-operator (an operator realizable in a finite automaton) of weight $\leq k$.

Remark. An analogous assertion, but without an estimate for the weight of the solution, is also valid for open $\hat{X}\hat{Y}\mathfrak{A}(X, Y)$.

8°. Let an arbitrary formula $\mathfrak{A}(X, Y)$ be given. Naturally, the following mass problems arise (formulated by Church ([3]) for formulas without predicate quantifiers):

I. To determine whether $\mathfrak{A}$ has solutions realizable in finite automata.

II. If so, to find all such solutions.

Problem II needs refinement. In this connection we introduce the following concept:

Definition. An L-operator $Y = T(X, U, V)$ is a general L-solution of the formula $\mathfrak{A}(X, Y)$ if the following conditions are fulfilled: 1) every L-solution of $\mathfrak{A}(X, Y)$ can be obtained from T by replacing U, V with appropriate L-operators $U = T_u(X)$, $V = T_v(X)$; 2) under any replacement in T of the variables U, V by L-operators $T_u(X), T_v(X)$, an L-solution of $\mathfrak{A}(X, Y)$ is obtained. If in this definition one everywhere replaces the terms "L-solution," "L-operator" by the terms "solution without anticipation," "operator without anticipation," then one obtains the definition of the notion of a general "D-solution."

Theorem 5. Whatever the formula $\mathfrak{A}$ such that $\hat{X}\hat{Y}\mathfrak{A}(X, Y)$ is closed, the following alternative holds: 1) $\mathfrak{A}$ has no solutions without anticipation at all; 2) there exists a general L-solution for $\mathfrak{A}$ with weight not exceeding its degree of homogeneity; it is also a general D-solution.

The proof consists in describing a construction from which, given a decision algorithm for I, one could extract an algorithm (for a general solution) which,

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when applied to the formula $\mathfrak{A}$ from the theorem, determines whether 1) or 2) is true, and in case 2) constructs a general L-solution.

Corollary. If $\mathfrak{A}$ has the form $(*)$, and every closed $\hat{X}\hat{Y}\mathfrak{A}(XY)$ is representable in this form, then an algorithm for the general solution exists, for in this case, instead of an -algorithm, one may be satisfied with Putnam' s algorithm ([6]).

Church developed an algorithm for obtaining a general -solution as applied to the class of formulas of the form (t) [quantifier-free formula]. Thus, our construction covers a substantially broader class of formulas.

9°. Robinson ([7]) conjectures that every function with natural arguments and natural values which is definable in is also definable in $'$([2]).

Let us note that the validity of this hypothesis follows from the fact that every set of special predicates ([4]) definable in is definable in $'$.

Proof correction note. After this work had been completed, the author learned that the paper ([9]) contains a result related to the corollary of Theorem 5; however, the notion of a general solution is absent from that paper.

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Received
29 IV 1961

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* In paper ([2]), on p. 320, line 19, instead of the printed
$(Et)(\tau)\,\bar{X}(\tau)$, one should read
$(Et)(\tau)\,\bar{X}_{\tau>t}(\tau)$; on p. 320, line 5 from the bottom, instead of $\cdots\&(\tau)\ \cdots$, one should read $\cdots\&_{\tau>t}(\tau)\ \cdots$

Note: Figure translations are in progress. See original paper for figures.

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