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## Abstract

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# APPROXIMATIVE COMPACTNESS AND CHEBYSHEV SETS

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In this note we continue the study of Chebyshev sets in Banach spaces, begun in our papers ([1–3]). Here we give a necessary and sufficient condition for a Chebyshev set lying in a uniformly convex and smooth Banach space to be convex. We apply this result to the study of the approximative properties of the set of rational fractions with prescribed degrees of numerator and denominator in the spaces $L_p$ $(p > 1)$.

In the formulations and proofs we make essential use of the new concept of approximative compactness of sets in Banach spaces, to which the first part of the note is devoted. We were compelled to formulate this concept because bounded compactness, which we considered in note ([3]), is insufficient for studying the Chebyshev properties of the set of rational fractions.

**1. Approximatively compact sets in Banach spaces.** Let $X$ be a real Banach space and let $M$ be some subset of it. If $x \in X$, $y_n \in M$ $(n = 1, 2, ...)$, and $\lim_{n\to\infty} \rho(x, y_n) = \rho(x, M)$, then we shall say that the sequence $\{y_n\}$ is **minimizing** for $x$ in $M$.

**Definition.** A set $M \subset X$ will be called **approximatively compact** if, for every $x \in X$, every minimizing sequence of elements $y_n \in M$ is compact in $M$.

The class of all approximatively compact sets is quite broad. It is clear that every boundedly compact set is approximatively compact. The converse, generally speaking, is false, since, for example, every closed convex set in a uniformly convex Banach space is approximatively compact.

It is easy to see that approximatively compact sets are closed. Moreover, every approximatively compact set is a set of existence, i.e., for every element $x \in X$ the lower bound

$$\inf_{y\in M} \rho(x, y)$$

is attained at some element $y_0 \in M$ (not necessarily unique).

We shall indicate a simple criterion for approximative compactness of a set. A set $M \subset X$ will be called **sequentially weakly closed** if every point of the space that is the weak limit of some sequence of elements $y_n \in M$ belongs to $M$. This definition is topological if the space $X$ is reflexive (see ([4])).

**Lemma 1.** *Let $X$ be a uniformly convex Banach space. If a set $M \subset X$ is sequentially weakly closed, then it is approximatively compact.*

A number of propositions proved by us earlier for boundedly compact sets carry over to approximatively compact sets. For example:

**Lemma 2** (on removal of a ball). *Let $M$ be an approximatively compact set in a Banach space $X$; let $\bar{E}$ be a closed ball supporting $M$ at a single point $x_0$, and let $y_0$ belong to the open ball $E$. Put $e = y_0 - x_0$. Then there exists a number $\lambda_0 > 0$ such that the translated ball*

$$\bar{E}' = \bar{E} + \lambda e$$

*does not intersect $M$ for all $\lambda,\ 0 < \lambda \leq \lambda_0$.*

Moreover, approximatively compact sets have a number of useful properties not possessed by boundedly compact sets. Let us note the following proposition:

**Lemma 3** (on $a$-extensions). *Let $X$ be a uniformly convex Banach space and let $M$ be its approximatively compact subset. Then for any $a > 0$ the closed $a$-extension $M_a$ of the set $M$ is approximatively compact.*

Recall that by the closed $a$-extension of $M$ we mean the set $M_a$ of all points $x \in X$ for which $\rho(x, M) \leq a$.

**2. Approximatively compact Chebyshev sets.** A set $M \subset X$ is called **Chebyshev** if for every $x_0 \in X$ there exists a unique point $y_0 \in M$ for which

$$\rho(x_0, M) = \rho(x, y).$$

We call a set $M \subset X$ a **sun** in $X$ if it is a set of existence and has the following property: let $x$ be an arbitrary point not belonging to $M$, and let $y \in M$ be such that $\rho(x, M) = \rho(x, y)$; then for every point $z$ of the ray issuing from $y$ and passing through $x$,

$$\rho(z, M) = \rho(z, y).$$

Let $M \subset X$ and $a > 0$ be given; by the $a$-**hull** of the set $M$ we mean the intersection of the complements of all open balls $E_a$ of radius $a$ that do not intersect $M$.

We call a set $M \subset X$ $a$-**convex** if it coincides with its $a$-hull.

**Theorem 1.** *In a uniformly convex space $X$, for an approximatively compact set the property of being a sun in $X$ is equivalent to the Chebyshev property.*

**Theorem 2.** *An approximatively compact set in a uniformly convex space is Chebyshev if and only if each of its closed $b$-extensions is $a$-convex for every $a > 0$.*

The proofs of these theorems, as well as the proofs of analogous propositions formulated in our note ([3]) for boundedly compact sets, are based on Theorem 1 and Lemma 2 of note ([3]), and also on Lemmas 2 and 3 of the present note.

If, in addition to the uniform convexity of $X$, we assume that $X$ is smooth (i.e., that at every boundary point $x_0$ of a ball $\bar{E} \subset X$ there exists a unique hyperplane supporting $\bar{E}$ at $x_0$), then every sun in $X$ is convex. Hence, from Theorem 1 the following easily follows:

**Theorem 3** (main). *Let $X$ be a uniformly convex and smooth Banach space. In order that a Chebyshev set $M \subset X$ be convex, it is necessary and sufficient that it be approximatively compact.*

The question of whether every Chebyshev set lying, for example, in a Hilbert space is approximatively compact has not been solved by us.

3. **Rational fractions in the spaces $L_p$** $(p > 1)$. Denote by $R_{m,n}$ the set of all rational fractions of the form

$$R(x) = \frac{P(x)}{Q(x)} = \frac{p_0 + p_1 x + \cdots + p_{mx}^m}{q_0 + q_1 x + \cdots + q_{nx}^n},$$

belonging to the space $L_p[0,1]$, $p > 1$. If $n \geq 2$, then the set $R_{m,n}$ is not boundedly compact in $L_p$. Indeed, set

$$R_k(x) = \frac{b_k^{2-\frac{1}{p}}}{x^2 + b_k^2} \qquad (k = 1, 2, \ldots),$$

where $b_k = 4^{-pk}$. Then $\|R_k(x)\|_{L_p} \leq A$, where $A$ is an absolute constant, and

$$\|R_k(x) - R_l(x)\|_{L_p} \geq \tfrac{1}{4}, \qquad k \neq l.$$

Consequently, the bounded set $\{R_k(x)\}$ in $L_p$ is not compact.

It can be shown that for all $m \geq 0,\ n \geq 0$ the set $R_{m,n}$ is sequentially weakly closed in $L_p$ $(p > 1)$ and, hence, approximatively compact. Since, moreover, for $n \geq 1$ the set $R_{m,n}$ is not convex, it follows from the main theorem that for any $m \geq 0,\ n \geq 1$ the set $R_{m,n}$ in the space $L_p$ $(p > 1)$ is a set of existence, but is not a Chebyshev set.

This fact reveals the special position of Chebyshev' s theorem, according to which $R_{m,n}$ is a Chebyshev set in the space of real continuous functions $C[0,1]$.

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*Note: Figure translations are in progress. See original paper for figures.*