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V. I. Gavrilov

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Abstract

Full Text

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Mathematics

In the present note we consider the question of the structure of the set of angular boundary values of meromorphic functions in the unit disk $|z| < 1$ that are normal there in the sense of Lehto and Virtanen ([1]). In the proofs of our assertions we shall rely essentially on the results of Collingwood and Cartwright ([2]).

1. In the terminology of ([2]) one can formulate the following theorem:

Theorem 1. *If the function $w = f(z)$ is meromorphic and normal in the disk $|z| < 1$, then:*

a) $\Gamma(f, \theta_1 < \theta < \theta_2) = \Gamma_P(f, \theta_1 < \theta < \theta_2)$ on every arc (θ_1, θ_2) of the unit circle $|z| = 1$ ([3]);

b) if $a \in \Gamma_P(f, e^{i\theta})$, then $\Gamma_P(f, e^{i\theta})$ contains no other values and $e^{i\theta} \in F(f)$ ([1]).

Theorem 1, establishing a one-to-one correspondence between the sets $\Gamma(f)$ and $F(f)$ considered for a normal meromorphic function $f(z)$, says nothing, however, about whether these sets contain even a single point. Moreover, an example is known ([1]) of a normal meromorphic function $\varphi(z)$ in $|z| < 1$ for which the sets $\Gamma(\varphi)$ and $F(\varphi)$ are empty. In this case the set $CR(\varphi)$ was also empty for the function $\varphi(z)$.

If, however, for a normal meromorphic function $f(z)$ in $|z| < 1$ the set $CR(f)$ is nonempty, then the sets $\Gamma(f)$ and $F(f)$ are also nonempty, as is indicated by Corollary 9.3 of ([2]).*

2. **Theorem 2.** *If the meromorphic function $w = f(z) \not\equiv \text{const}$ is normal in the disk $|z| < 1$ and the set $CR(f)$ is nonempty: $a \in CR(f)$, then:*

a) on every arc (θ_1, θ_2), $0 \le \theta_1 < \theta_2 \le 2\pi$, of the circle $|z| = 1$, the set $F(f)$ is nonempty;

b) if, moreover, on (θ_1, θ_2) $\operatorname{mes} F(f) = 0$, then there exists a point $e^{i\theta_0}$, $\theta_1 < \theta_0 < \theta_2$, $e^{i\theta_0} \in F(f)$, at which the function $w = f(z)$ has the angular boundary value equal to a, and $F(f)$ will be a set of first category on (θ_1, θ_2).

Proof. Suppose the contrary, and let the arc $(|\theta| < \eta)$, $\eta > 0$, contain no points of the set $F(f)$. Then, by Lemma 11 of ([1]): 1) either $a \in CC(f,1)$, 2) or $a \in R(f,1)$, 3) or $a \in \Phi(f,1)$.

If $a \in CC(f,1)$, then, by Theorem 14 of ([2]), the arc $(|\theta| < \eta)$ would contain points of the set $F(f)$. The second case is also impossible, by the hypothesis.

If $a \in \Phi(f,1)$, then there exists a sequence of mutually nonintersecting continuous arcs $\{\gamma_n\}$, whose endpoints tend to the points $e^{\pm i\delta}$, $0 < \delta < \eta$, on which $f(z)$ tends uniformly to a as $|z| \to 1$.

Draw through the point $z = 1$ a diameter L_0 of the disk $|z| < 1$, which intersects the arcs $\{\gamma_n\}$ at the points $\{z_n^0\}$. If the non-Euclidean distances $\rho(z_n^0, z_{n+1}^0)$

* This assertion was announced in ([4]); for a proof by a method different from the present one, see also ([3]).

between neighboring points of the sequence are uniformly bounded by some constant M, then, by the lemma from ([3]), $a \in \Gamma_p(f,1)$.

In the general case, to each pair of neighboring points z_k^0, z_{k+1}^0 we adjoin a finite number of points

$$z_k^\nu, \quad \nu = 1, \dots, p_k, \quad z_k^1 = z_k^0, \dots, z_k^{p_k} = z_{k+1}^0,$$

lying on L_0, so that

$$\rho\left(z_k^\nu, z_k^{\nu+1}\right) = 1, \quad \nu = 1, 2, \dots, p_k.$$

Through the points z_k^ν draw curves γ_k^ν so that

$$\gamma_k^0 = \gamma_k^1, \dots, \gamma_k^{p_k} = \gamma_{k+1}^0, \quad \rho\left(\gamma_k^\nu, \gamma_k^{\nu+1}\right) = 1.$$

To the radius L_0 draw equidistants Λ_s, $s = 1, 2, \dots, p_k$, for which

$$\rho(L_0, \Lambda_s) = s.$$

Let the points of intersection of the curves Λ_s and γ_k^ν be

$$z_{k,s}^\nu, \quad \nu = 1, \dots, p_k; \quad s = 1, \dots, p_k; \quad k = 1, 2, \dots .$$

Denote the aggregate of points

$$\{z_k^0;\ z_{k,2}^2;\ z_{k,3}^3; \dots;\ z_{k+1,p_k}^0;\ z_{k+1,p_k-1}^0; \dots;\ z_{k+1}^0\}$$

by E_k. Let $E = \bigcup_k E_k$; by construction, E is a sequence of points $\{z_m\}$, whose limit points lie on $|z| = 1$, and

$$\rho(z_m, z_{m+1}) \le M$$

for all m. Since, by the lemma from ([3]),

$$\lim_{\nu\to\infty} f\left(z^{\nu}_{k,s}\right) = a$$

for every s, it follows, in view of the choice of E_k, that

$$\lim_{m\to\infty} f(z_m) = a.$$

If now the set of limit points of the sequence $\{z_m\}$ consists of a single point $e^{i\theta_0}$, then $a \in \Gamma_p(f, e^{i\theta_0})$, $|\theta_0| < \eta$; if, however, this set contains more than one point, then, by Theorem 3 from ([3]), $f(z) \equiv a$.

The assertion of part b) of Theorem 2 is an immediate consequence of Theorem 16 of the work ([2]), of the constructions carried out in the proof of part a), and of Theorem 4 of the work ([5]).

Remark 1. The construction used in proving the impossibility of case 3), in essence, shows that for a normal meromorphic function in $|z| < 1$ the set of Plessner points cannot fill any arc on $|z| = 1$ completely.

3. We give some consequences of the theorems obtained above.

Theorem 2 and the uniqueness theorem of N. N. Luzin–I. I. Privalov immediately lead to the following assertion:

Theorem 3. *If, for a normal meromorphic function $w = f(z) \not\equiv \text{const}$ in the disk $|z| < 1$, the set $CR(f)$ contains at least one point, then the set $\Phi(f)$ is empty.*

If one slightly modifies the construction carried out for the proof of the impossibility of case 3) in Theorem 2, and, instead of the lemma from ([3]), uses Lemma 1 from ([4]), instead of Theorem 3 from ([3]) and Theorem 1 from ([4])–Theorems 4 from ([3]) and 2 from ([4]), then one can show that the following theorem is true:

Theorem 4. *If $w = f(z) \not\equiv \text{const}$ is a meromorphic normal function in the disk $|z| < 1$, then the set $\Phi(f)$ is empty.*

Theorems 3 and 4 may be regarded as generalizations of the Koebe–Gross theorem ([2]).

Remark 2. Just as in ([3]), one can show that the theorems given above remain valid also in the general case of normal quasiconformal functions.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.