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1961

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Abstract

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MATHEMATICS

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SOME PROPERTIES OF DIFFUSION PROCESSES DEPENDING ON A PARAMETER

(Presented by Academician A. N. Kolmogorov, 21 I 1961)

Consider the stochastic equation

$$x_t(a,\omega) - x_s(a,\omega) =$$

$$= \int_s^t \sigma(u,a,x_u(a,\omega))\,d\xi_u(\omega) + \int_s^t m(u,a,x_u(a,\omega))\,du. \tag{1}$$

In equation (1), $\xi_u(\omega) = (\xi_u^1(\omega), \xi_u^2(\omega), \dots, \xi_u^n(\omega))$ is an n-dimensional Wiener process defined on the probability space $(\Omega, \mathfrak{M}, P)$; $\sigma(u,a,x) = \{\sigma_j^i(u,a,x)\}_{i,j=1}^n$ is a matrix; $m(u,a,x) = (m^1(u,a,x), m^2(u,a,x), \dots, m^n(u,a,x))$ is an n-dimensional vector. The integrals on the right-hand side of equation (1) are understood as stochastic integrals (see ([1])). Suppose that the elements of the matrix $\sigma(u,a,x)$ and of the vector $m(u,a,x)$ uniformly satisfy a Lipschitz condition in x. Then it is proved that there exists a random Markov function $x_t(a,\omega)$, taking values in the n-dimensional Euclidean space R^n and satisfying equality (1) with probability 1. The parameter $a = (a_1, a_2, \dots, a_m)$ takes values in some domain A in R^m.

In the present note we give a number of results concerning the continuity and differentiability of $x_t(a,\omega)$ with respect to a.

Theorem 1. *Suppose that there exists a constant $C < \infty$ such that for any $x, y \in R^n$, $a, \beta \in A \subseteq R^m$, $u \in [0,T]$, $T < \infty$,*

$$\sum_{i,j=1}^n \left|\sigma_j^i(u,a,x) - \sigma_j^i(u,\beta,y)\right| + \sum_{i=1}^n \left|m^i(u,a,x) - m^i(u,\beta,y)\right| \leq$$

$$\leq C(\|a-\beta\| + \|x-y\|)^*.$$

Suppose also that $x_0(a,\omega)$ is continuous in $a \in A$ for almost all ω. Then there exists a random function $x_t(a,\omega)$, satisfying equation (1) and, with probability 1, continuous in $(t,a) \in [0,T] \times A$.

Theorem 2. *Suppose that* $\sigma_j^i(u,a,x)$, $m^i(u,a,x)$ *have continuous bounded derivatives with respect to* a_p, x^r $(i,j,r = 1,2,\dots,n;\ p = 1,2,\dots,m)$ *up to order* $k+1$ *inclusive. Suppose also that* $x_0(a,\omega)$ *and* $d^l x_0(a,\omega)/\partial a_1^{l_1} \cdots \partial a_m^{l_m}$, $l_1 + l_2 + \dots + l_m = l \le k+1$, *exist for almost all* ω, *are bounded and continuous.*

Then, for almost all ω *and all* $l_1 + l_2 + \dots + l_m = l \le k$, *there exist derivatives continuous in* (t,a),

$$\partial^l x_t(a,\omega)/\partial a_1^{l_1} \partial a_2^{l_2} \cdots \partial a_m^{l_m}.$$

If the requi– to require the existence of $d^l x_0(a,\omega)/\partial a_1^{l_1} \partial a_2^{l_2} \cdots \partial a_m^{l_m}$ only in the mean-square sense* for all $l_1 + l_2 + \dots + l_m = l \le k+1$, while keeping the same requirements for $\sigma(u,a,x)$ and $m(u,a,x)$ as above, then $\partial^l x_t(a,\omega)/\partial a_1^{l_1} \cdots \partial a_2^{l_2} \partial a_m^{l_m}$ will also exist in the mean-square sense for all $l_1, l_2, \dots, l_m$, $l_1 + l_2 + \dots + l_m = l \le k$. The family of random functions $\partial^l x_t(a,\omega)/\partial a_1^{l_1} \cdots \partial a_m^{l_m}$, $l_1 + l_2 + \dots + l_m = l \le k$, satisfies the following system of stochastic equations:

$$\frac{\partial^l x_t(a,\omega)}{\partial a_1^{l_1} \partial a_2^{l_2} \cdots \partial a_m^{l_m}} = \frac{\partial^l x_0(a,\omega)}{\partial a_1^{l_1} \partial a_2^{l_2} \cdots \partial a_m^{l_m}} + \int_0^t \frac{\widetilde{\partial^l} \sigma(u,a,x_u(a,\omega))}{\partial a_1^{l_1} \partial a_2^{l_2} \cdots \partial a_m^{l_m}} \, d\xi_u(\omega) + \int_0^t \frac{\widetilde{\partial^l} m(u,a,x_u(a,\omega))}{\partial a_1^{l_1} \partial a_2^{l_2} \cdots \partial a_m^{l_m}} \, du. \tag{2}$$

Here

$$\frac{\widetilde{\partial f}(a_1, a_2, \dots, a_m; x^1(a), x^2(a), \dots, x^n(a))}{\partial a_k} = \frac{\partial f}{\partial a_k} + \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial x^i}{\partial a_k};$$

$$\frac{\widetilde{\partial^2 f}}{\partial a_i \partial a_j} = \frac{\tilde{\partial}}{\partial a_i} \left(\frac{\widetilde{\partial f}}{\partial a_j} \right).$$

The following follows from Theorem 2.

Theorem 3. Let $x_t^x(\omega)$ satisfy the stochastic equation

$$x_t^x(\omega) = x + \int_0^t \sigma(u, x_u^x(\omega)) \, d\xi_u(\omega) + \int_0^t m(u, x_u^x(\omega)) \, du. \tag{3}$$

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* If $z_i = (z_i^1, z_i^2, \dots, z_i^k)$, $i = 1,2$, then $\|z_1 - z_2\| = \left(\sum_{j=1}^k |z_1^j - z_2^j|^2 \right)^{1/2}$.

Then, if $\sigma(u, x)$ and $m(u, x)$ have bounded continuous derivatives up to order $k+1$, inclusive, with respect to x^r, $r = 1, 2, \dots, n$, then for all $l \leq k$, for almost all ω, the derivatives $\partial^l x_t^x(\omega)/\partial(x^1)^{l_1} \cdots \partial(x^n)^{l_n}$, $l_1 + l_2 + \cdots + l_n = l$, exist. These derivatives also exist in the mean-square sense.

For the proof of the assertions formulated, we shall need the following generalization of A. N. Kolmogorov' s well-known theorem on the continuity of sample functions of a process.

Theorem. Let $x_\mu(\omega)$ be a separable random field,[**] defined for $\mu \in R^m$ and taking values in n-dimensional Euclidean space R^n. Then, in order that $x_\mu(\omega)$ be continuous in μ with probability 1, it is sufficient that, for some $\gamma > 0$, $\varepsilon > 0$, the inequality

$$M\|x_\mu(\omega) - x_{\mu'}(\omega)\|^\gamma \leq C\|\mu - \mu'\|^{m+\varepsilon}$$

hold.

The proof of this theorem is carried out analogously to the case $m = 1$ (see (1), p. 576).

With the aid of the change-of-variables formula in stochastic integrals (2), when the conditions of Theorem 1 are fulfilled, one can prove the following inequality, valid for positive integers n:

$$M\|x_t(a, \omega) - x_s(\beta, \omega)\|^{2n} \leq C_n(\|a - \beta\|^{2n} + |t - s|^n); \tag{4}$$

$$\alpha, \beta \in A, \quad t,\ s \in [0, T].$$

From the last inequality, by virtue of the above-formulated generalization of A. N. Kolmogorov' s theorem, Theorem 1 follows. (Independently, Theorem 1 was proved by I. V. Girsanov.)

Let us explain the proof of Theorem 2 in the case $n = m = l = 1$. The random functions $x_t(\beta, \omega)$ and $x_t(\beta', \omega)$ are solutions of equation (1) for $\alpha = \beta$ and $\alpha = \beta'$, respectively. The function

$$y_t^{\beta\beta'}(\omega) = \frac{x_t(\beta, \omega) - x_t(\beta', \omega)}{\beta - \beta'}$$

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[*] That is, the limit in the definition of the derivative is understood as a limit in the mean-square sense.

[**] Under natural assumptions, for every field $x_\mu(\omega)$ there exists an equivalent separable field $\tilde{x}_\mu(\omega)$.

satisfies the equation

$$
\begin{aligned}
y_t^{\beta\beta'}(\omega) = y_0^{\beta\beta'}(\omega) + \int_0^t \Bigg[&\frac{\sigma(u,\beta,x_u(\beta,\omega)) - \sigma(u,\beta',x_u(\beta,\omega))}{\beta\beta'} + \\
&+ \frac{\sigma(u,\beta',x_u(\beta,\omega)) - \sigma(u,\beta',x_u(\beta',\omega))}{x_u(\beta,\omega) - x_u(\beta',\omega)}\, y_u^{\beta\beta'}(\omega) \Bigg] d\xi_u(\omega) + \\
+ \int_0^t \Bigg[&\frac{m(u,\beta,x_u(\beta,\omega)) - m(u,\beta',x_u(\beta,\omega))}{\beta - \beta'} + \\
&+ \frac{m(u,\beta',x_u(\beta,\omega)) - m(u,\beta',x_u(\beta',\omega))}{x_u(\beta,\omega) - x_u(\beta',\omega)}\, y_u^{\beta\beta'}(\omega) \Bigg] du.
\end{aligned}
\tag{5}
$$

Equation (5), together with equation (1) taken for $\alpha = \beta$ and $\alpha = \beta'$, forms a system of stochastic equations for the random function

$$
z_t(\beta,\beta',\omega) = (x_t(\beta,\omega),\, x_t(\beta',\omega),\, y_t^{\beta\beta'}(\omega)).
$$

Using the fact that $\sigma(u,\alpha,x)$ and $m(u,\alpha,x)$ are differentiable, it is not difficult to prove that the coefficients of equation (5) satisfy conditions ensuring the existence of a solution of the system of stochastic equations for the function $z_t(\beta,\beta',\omega)$. To prove the existence of the derivative

$$
dx_t(\beta,\omega)/d\beta = \lim_{\beta'\to\beta} y_t^{\beta\beta'}(\omega),
$$

it remains to verify that $z_t(\beta,\beta',\omega)$ is, with probability 1, continuous in (t,β,β') for $t \in [0,T]$, $\beta,\beta' \in A$. The latter assertion follows from the generalized theorem of A. N. Kolmogorov by means of inequalities analogous to (4). From simple estimates for stochastic equations it follows that $dx_t(\beta,\omega)/d\beta$ satisfies the equation

$$
\begin{aligned}
\frac{dx_t(\beta,\omega)}{d\beta} = \frac{dx_0(\beta,\omega)}{d\beta} &+ \int_0^t \left[\frac{\partial\sigma(u,\beta,x_u(\beta,\omega))}{\partial\beta} + \frac{\partial\sigma(u,\beta,x_u(\beta,\omega))}{\partial x}\frac{dx_u(\beta,\omega)}{d\beta} \right] d\xi_u(\omega) + \\
&+ \int_0^t \left[\frac{\partial m(u,\beta,x_u(\beta,\omega))}{\partial\beta} + \frac{\partial m(u,\beta,x_u(\beta,\omega))}{\partial x}\frac{dx_u(\beta,\omega)}{d\beta} \right] du,
\end{aligned}
\tag{6}
$$

which is the limit of (5) as $\beta' \to \beta$. From inequalities similar to (4), it is not difficult to derive that $y_t^{\beta\beta'}(\omega)$ converges to $dx_t(\beta,\omega)/d\beta$ also in mean square.

Remark 1. The coefficients of the equations for higher derivatives, generally speaking, grow faster than $\|x\|$, and therefore the existence of a solution of system (2) has to be proved separately.

Remark 2. With the help of certain additional constructions, in Theorem 2 one can dispense with the requirement that the derivatives be bounded: it is sufficient that, for some $N < \infty$, they grow no faster than $\|x\|^N$. If one is not concerned with convergence in mean square, then it is sufficient to require of

$x_0(\alpha, \omega)$ the existence and continuity, for almost all ω, of all partial derivatives with respect to α_p, $p = 1, 2, \dots, m$, up to order $k+1$ inclusive.

Theorem 3 is strengthened analogously.

Remark 3. If $x_t^\alpha(a, \omega)$ is a one-dimensional random Markov function satisfying equation (3), then the equation for $dx_t^\alpha(\omega)/d\alpha$ has the form

$$\frac{dx_t^\alpha(\omega)}{d\alpha} = 1 + \int_0^t \frac{\partial \sigma(u, x_u^\alpha(\omega))}{\partial x} \frac{dx_u^\alpha(\omega)}{d\alpha} \, d\xi_u(\omega) + \int_0^t \frac{\partial m(u, x_u^\alpha(\omega))}{\partial x} \frac{dx_u^\alpha(\omega)}{d\alpha} \, du.$$

This equation can be solved in explicit form:

$$\frac{dx_t^a(\omega)}{da} = \exp\left\{ \int_0^t \frac{\partial \sigma(u, x_u^a(\omega))}{\partial x} \, d\xi_u(\omega) + \int_0^t \frac{\partial m(u, x_u^a(\omega))}{\partial x} \, du - \frac{1}{2} \int_0^t \left[\frac{\partial \sigma(u, x_u^a(\omega))}{\partial x} \right]^2 du \right\}.$$

Let us indicate some applications of the results obtained. Let $x_t^x(\omega)$ be the solution of the stochastic equation (3); $x_0^x(\omega) \equiv x$; and let $f(x)$ be a k-times continuously differentiable function. Then, if the conditions of Theorem 3 are satisfied, the function $u(x, t) = Mf(x_t^x(\omega))$ has continuous partial derivatives with respect to x^r, $r = 1, 2, \dots, n$, up to and including order k. On the other hand, it is known that if the function $u(x, t) = Mf(x_t^x(\omega))$ possesses continuous partial derivatives up to and including second order with respect to x^r, then

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n b_j(t, x) \frac{\partial u}{\partial x^i}, \tag{7}$$

where

$$a_{ij}(t, x) = \sum_{k=1}^n \sigma_k^i(t, x) \sigma_k^j(t, x), \qquad b_i(t, x) = m^i(t, x), \qquad i, j = 1, 2, \dots, n.$$

Thus, if $\sigma_j^i(t, x)$, $m^i(t, x)$ have continuous bounded partial derivatives with respect to x^r, $r = 1, 2, \dots, n$, up to and including order $k+1$, and $f(x)$ has continuous partial derivatives up to and including order k, then the solution of the Cauchy problem for equation (7) also has continuous partial derivatives with respect to x^r up to and including order k (the matrix $a_{ij}(t, x)$ may also be degenerate).

In conclusion we formulate the following theorem, whose proof is based on Theorem 2.

Theorem 4. *Let $x_t(a,\omega)$ be a random function satisfying equation (1), the coefficients of which possess continuous bounded partial derivatives up to and including second order. Denote by*

$$\tau_D^a(\omega) = \inf\{t : x_t(a,\omega) \notin D\},$$

where D is a domain in R^n whose boundary Γ has a continuously rotating normal. Then, if

$$\det \left|\{\sigma_j^i(t,x)\}_1^n\right| \neq 0$$

for $x \in \Gamma$ and $t \geq 0$, then with probability 1

$$\frac{\partial \tau_D^a(\omega)}{\partial a_i} = 0, \qquad i = 1, 2, \dots, m,$$

for all a, except for some set $\Lambda(\omega) \in R^m$ of Lebesgue measure zero.

The authors express their gratitude to E. B. Dynkin for posing the problem and for a number of substantial comments.

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Received
14 I 1961

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Note: Figure translations are in progress. See original paper for figures.

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