S. K. GODUNOV

In this note it will be shown that a nonlinear parabolic system of equations of the form

1961

Abstract

Full Text

MATHEMATICS

S. K. GODUNOV

AN EXAMPLE OF NONUNIQUENESS FOR A NONLINEAR PARABOLIC SYSTEM

(Presented by Academician I. G. Petrovskii, 7 X 1960)

In this note it will be shown that a nonlinear parabolic system of equations of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left[a(u,v)\frac{\partial u}{\partial x}\right], \\ \frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} \end{aligned} \tag{1}$$

does not necessarily have a unique solution satisfying the initial data

$$u(x,0) = v(x,0) = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases} \tag{2}$$

After acquainting himself with this work, A. N. Tikhonov drew my attention to the fact that, if boundedness of the solution is not required, then uniqueness will fail even in the case of the equation $\partial u/\partial t = \partial^2 u/\partial x^2$. In the example constructed by me both solutions are bounded.

The example of nonuniqueness that will be constructed uses a discontinuous function $a(u,v)$; however, this is apparently inessential. It is easy to imagine a process of smoothing the discontinuities in $a(u,v)$ that does not obstruct the implementation of the idea of the construction described. We begin by giving solutions with the initial data (2) for two auxiliary systems

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2}, & \frac{\partial v_1}{\partial t} &= \frac{\partial^2 v_1}{\partial x^2}; \\ \frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial x}\left[k\left(\frac{x}{\sqrt{t}}\right)\frac{\partial u_2}{\partial x}\right], & \frac{\partial v_2}{\partial t} &= \frac{\partial^2 v_2}{\partial x^2}; \end{aligned}$$

$$K\left(\frac{x}{\sqrt{t}}\right) = \begin{cases} 0.6944, & \text{if } \frac{1}{4} < \left|x/\sqrt{t}\right| < \frac{1}{2}, \\ 1, & \text{otherwise.} \end{cases}$$

Fig. 1

Figure 1: Fig. 1

These solutions have the form

$$u_1 = v_1 = v_2 = \Phi\left(\frac{\xi}{2}\right), \qquad \xi = \frac{x}{\sqrt{t}}, \qquad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\zeta^2}\, d\zeta;$$

$$u_2(\xi) = \begin{cases} 0.91373\,\Phi\left(\frac{1}{2}\xi\right) - 0.086263, & \text{for } \xi \le -\frac{1}{2}, \\ 1.22399\,\Phi\left(\frac{3}{5}\xi\right) + 0.063484, & \text{for } -\frac{1}{2} \le \xi \le -\frac{1}{4}, \\ 1.01301\,\Phi\left(\frac{1}{2}\xi\right), & \text{for } -\frac{1}{4} \le \xi \le \frac{1}{4}, \\ 1.22399\,\Phi\left(\frac{3}{5}\xi\right) - 0.063484, & \text{for } \frac{1}{4} \le \xi \le \frac{1}{2}, \\ 0.91373\,\Phi\left(\frac{1}{2}\xi\right) + 0.086263, & \text{for } \frac{1}{2} \le \xi. \end{cases}$$

Let us depict in the u, v plane the curves given parametrically by the equations

$$u = u_1(\xi), \qquad v = v_1(\xi); \tag{I}$$

$$u = u_2(\xi), \qquad v = v_2(\xi). \tag{II}$$

For clarity, we have drawn these curves in a somewhat transformed plane: along the abscissa axis we plot $u + v$, and along the ordinate axis $u - v$ (see Fig. 1). Curve (I) is a segment of the abscissa axis, marked with hatching; curve (II) is shown by a solid line. The portions of curve (II) corresponding to those values of $x/\sqrt{t}$ for which $K(x/\sqrt{t}) = 0.6944$ are indicated by a double line. Put $a(u, v) = 0.6944$ in the hatched regions. These regions must intersect curve (II) only along the portions marked by the double line, and must not intersect curve (I) at all. Outside the hatched regions put $a(u, v) = 1$.

Fig. 1

It is obvious that both pairs of functions $[u_1(\xi), v_1(\xi)]$; $[u_2(\xi), v_2(\xi)]$, where $\xi = x/\sqrt{t}$, satisfy the initial data (2) and the system (1) with the coefficient $a(u, v)$ constructed in this way.

The possibility of a nonunique solution for nonlinear parabolic systems with discontinuous initial data became clear to me as a result of discussions with N. D. Vvedenskaya. I consider it my pleasant duty to express my gratitude to M. D. Takhtamysheva for the selection of numerical parameters for the example described.

Received
6 X 1960