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Abstract

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MATHEMATICS

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GEOMETRY OF AN ORDINARY SECOND-ORDER DIFFERENTIAL EQUATION WITH RESPECT TO THE CONFORMAL GROUP OF TRANSFORMATIONS OF TWO VARIABLES

(Presented by Academician P. S. Aleksandrov, 24 IV 1961)

§ 1. Formulation of the problem and the basic equation

In the present paper, by the invariant method proposed by G. F. Laptev ([1]) and A. M. Vasil' ev ([2]), a geometry is constructed for an ordinary second-order differential equation $d^2y/dx^2 = f(x, y, y')$ with respect to the infinite group of conformal transformations $\bar{x} = p(x, y)$, $\bar{y} = q(x, y)$; $p_x = q_y$, $p_y = -q_x$. In other words, objects invariantly associated with the given equation with respect to transformations of the conformal group are found, and invariant types of equations are singled out. By another method, É. Cartan ([3]) studied the analogous problem for the general analytic group.

The differential equation $y'' = f(x, y, y')$, where $y' = dy/dx$; $y'' = d^2y/dx^2$, may be regarded as a hypersurface in the 4-dimensional space (x, y, y', y''), in which the twice-prolonged conformal group acts. The structure equations of this group have the form (see ([4])):

$$\begin{aligned} D\omega^1 &= [\omega^1\omega^2] - [\tilde{\omega}^1\tilde{\omega}^2], \quad D\tilde{\omega}^1 = [\tilde{\omega}^1\omega^2] + [\omega^1\tilde{\omega}^2], \\ D\omega^2 &= [\omega^1\omega^3] - [\omega^1\tilde{\omega}^3], \quad D\tilde{\omega}^2 = [\tilde{\omega}^1\omega^3] + [\omega^1\tilde{\omega}^3]; \end{aligned} \tag{1}$$

$$\begin{aligned} D\omega^3 &= [\omega^2\omega^3] - [\tilde{\omega}^2\tilde{\omega}^3] + [\omega^1\omega^4] - [\tilde{\omega}^1\tilde{\omega}^4], \\ D\tilde{\omega}^3 &= [\tilde{\omega}^2\omega^3] + [\omega^2\tilde{\omega}^3] + [\tilde{\omega}^1\omega^4] + [\omega^1\tilde{\omega}^4]. \end{aligned} \tag{2}$$

Continuing once more, we obtain

$$\begin{aligned} D\omega^4 &= 2[\omega^2\omega^4] - 2[\tilde{\omega}^2\tilde{\omega}^4] + [\omega^1\omega^5] - [\tilde{\omega}^1\tilde{\omega}^5], \\ D\tilde{\omega}^4 &= 2[\tilde{\omega}^2\omega^4] + 2[\omega^2\tilde{\omega}^4] + [\tilde{\omega}^1\omega^5] + [\omega^1\tilde{\omega}^5]. \end{aligned} \tag{3}$$

The first integrals of completely integrable systems are: 1. $\omega^1 = 0,\ \tilde{\omega}^1 = 0$.
2. $\omega^1 = 0,\ \tilde{\omega}^1 = 0,\ \tilde{\omega}^1 = 0$.
3. $\omega^1 = 0,\ \tilde{\omega}^1 = 0,\ \tilde{\omega}^2 = 0,\ \tilde{\omega}^3 = 0$.

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They determine the space of points (x, y), the space of line elements (x, y, y'), and the space of second-order line elements (x, y, y', y''). The first integrals of the system $\omega^1,\ \tilde{\omega}^1,\ \omega^2,\ \tilde{\omega}^2,\ \omega^3,\ \tilde{\omega}^3$ may be chosen so that the following equalities hold:

$$\omega^1 = x'(dx + y'\,dy); \qquad \tilde{\omega}^1 = x'(dy - y'\,dx);$$

$$\tilde{\omega}^2 = \frac{1}{1+y'^2}(dy' - y''dx) + x''(dy - y'\,dx),$$

$$\tilde{\omega}^3 = \frac{1}{x'(1+y'^2)}\left[d\left(\frac{y''}{1+y'^2}\right) + x''dy'\right] + y'''dx + x'''dy,$$

$$\omega^2 = -\frac{dx'}{x'} - \frac{y'}{1+y'^2}\,dy' + x''dx + \left(\frac{y''}{1+y'^2} + x''y'\right)dy, \tag{4}$$

$$\omega^3 = -\frac{dx''}{x'} - \frac{y'}{x'(1+y'^2)}\left[d\left(\frac{y''}{1+y'^2}\right) + x''dy'\right] + x'''dx - y'''dy.$$

On the hypersurface $y'' = f(x, y, y')$ the forms $\omega^1,\ \tilde{\omega}^1,\ \tilde{\omega}^2,\ \tilde{\omega}^3$ are connected by one linear dependence, which by canonization is reduced to the form

$$\tilde{\omega}^3 = 0. \tag{5}$$

Now we have:

$$y'' = f(x, y, y'), \quad x'' = -\left(\frac{f}{1+y'^2}\right)y', \quad x''' = -\frac{f_y}{x'(1+y'^2)^2}, \quad y''' = -\frac{f_x}{x'(1+y'^2)^2}. \tag{6}$$

§ 2. **Invariants of the equation $y'' = f(x, y, y')$ with respect to the conformal group of transformations.** Differentiating equation (5) exteriorly and expanding by E. Cartan' s lemma, we obtain:

$$\tilde{\omega}^4 = u\omega^1 + v\tilde{\omega}^1 + k\omega^2, \quad \omega^4 = v\omega^1 + w\tilde{\omega}^1 + l\tilde{\omega}^2, \quad \omega^3 = k\omega^1 + l\tilde{\omega}^1 + m\tilde{\omega}^2. \tag{7}$$

Once more differentiating these equations and expanding by E. Cartan' s lemma:

$$\begin{aligned} du - 3u\omega^2 + \tilde{\omega}^5 \equiv 0, \qquad dv - 3v\omega^2 + \omega^5 \equiv 0, \qquad dw - 3w\omega^2 - \tilde{\omega}^5 \equiv 0,\\ dk - 2k\omega^2 \equiv 0, \qquad dl - 2l\omega^2 \equiv 0, \qquad dm - m\omega^2 \equiv 0 \quad (\mathrm{mod}\ \ \omega^1, \tilde{\omega}^1, \omega^2). \end{aligned} \tag{8}$$

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It is now clear that $m, k, l, u+w$ are relative invariants. If $m \neq 0$, then $I_1 = k/m^2,\ I_2 = l/m^2,\ I_3 = (u+w)/m^3$ are absolute invariants. Using the third equation (7), the equalities (4), (6), and the relation

$$D\omega^3 - [\omega^2\omega^3] \equiv (u+w)[\omega^1\tilde{\omega}^1]\ [\operatorname{mod} \tilde{\omega}^2],$$

we obtain, putting $\dfrac{f}{1+{y'}^2} = \hat{f}$:

$$m = \frac{1+{y'}^2}{x'}\hat{f}_{y'y'},$$

$$k = \frac{1}{{x'}^2(1+{y'}^2)}\left[\hat{f}_{xy'} + y'\hat{f}_{yy'} - \hat{f}_y + (1+{y'}^2)\hat{f}\hat{f}_{y'y'}\right],$$

$$l = \frac{1}{{x'}^2(1+{y'}^2)}\left[-y'\hat{f}_{xy'} + \hat{f}_{yy'} + \hat{f}_x - y'(1+{y'}^2)\hat{f}\hat{f}_{y'y'} + (1+{y'}^2)\hat{f}_y\hat{f}_{y'y'}\right], \tag{9}$$

$$\begin{aligned} u+w = \frac{1}{{x'}^3(1+{y'}^2)}\Bigg[&2\hat{f}_x\hat{f}_y - \frac{2y'}{1+{y'}^2}\hat{f}\hat{f}_x - \frac{2}{1+{y'}^2}\hat{f}\hat{f}_y + 2\hat{f}\hat{f}_{xy'} - 2\hat{f}_y\hat{f}_{y'y'} \\ &- 2y'\hat{f}_y\hat{f}_{xy'} + \frac{1}{1+{y'}^2}(\hat{f}_{xx} + \hat{f}_{yy}) - 2y'(1+{y'}^2)\hat{f}\hat{f}_y\hat{f}_{y'y'} \\ &+ (1+{y'}^2)^2\hat{f}_y^2\hat{f}_{y'y'} + (1+{y'}^2)\hat{f}^2\hat{f}_{y'y'}\Bigg]. \end{aligned}$$

In the case $m \neq 0$, by means of the form ω^2 we reduce m to unity: $m = 1$. Then $x' = (1+{y'}^2)\hat{f}_{y'y'}$. Substituting this value of x' into the equalities (9), we obtain expressions for $I_1,\ I_2,\ I_3$.

§ 3. **The space of linear elements of the affine connection associated with the equation** $y'' = f(x, y, y')$**.** Substituting the third equation (7) into equations (1) and making the replacement $\omega^1 = \theta^1,\ \tilde{\omega}^1 = \theta^2,\ \tilde{\omega}^2 = \theta_1^2 = -\theta_2^1,\ \omega^2 = \theta_1^1 = \theta_2^2$, we obtain:

$$D\theta^1 = [\theta^1\theta_1^1] + [\theta^2\theta_2^1], \qquad D\theta^2 = [\theta^1\theta_1^2] + [\theta^2\theta_2^2], \qquad D\theta_1^2 = -k[\theta^1\theta^2] + m[\theta^2\theta_1^2],$$

$$D\theta_1^1 = l[\theta^1\theta^2] + m[\theta^1\theta_1^2]. \tag{10}$$

These are the structure equations of the space of linear elements of the affine connection invariantly associated with our equation. If one calls a curve line a

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one-parameter family of linear elements tangent to a point curve described by their origin, then the equation of the geodesics has the form $\theta^2 = 0$; $\theta_1^2 = 0$. In view of (4) and (6), this gives $y' = dy/dx$, $y'' = d^2y/dx^2 = f(x, y, y')$. Consequently, the differential equation $y'' = f(x, y, y')$ is the equation of the geodesic lines of the associated space of linear elements of an affine connection.

Suppose now that $m = 0$. From (9) we obtain that the equation under consideration has the form

$$y'' = (1 + {y'}^2)\,[a(x, y)y' + b(x, y)]. \tag{11}$$

For such an equation the condition $k = 0$ has the form $a_x = b_y$, and the condition $l = 0$ has the form $a_y = -b_x$. We have obtained:

Theorem 1. An equation of the form $y'' = (1+{y'}^2)[a(x, y)y'+b(x, y)]$ preserves this form under an arbitrary transformation of the conformal group. If the original equation satisfied one of the conditions: 1) $a_x = b_y$; 2) $a_y = -b_x$; 3) $a_x = b_y,\ a_y = -b_x$, then the transformed equation also satisfies the corresponding condition.

If $m = 0,\ k = 0,\ l = 0$, then the adjoined space is flat. Therefore the family of its geodesics is carried by conformal transformations into a family of straight lines. Hence it follows:

Theorem 2. In order that a two-parameter family of lines in the plane be carried by a conformal transformation into a family of straight lines, it is necessary and sufficient that this family of curves be given by a differential equation of the form $y'' = (1 + {y'}^2)[a(x, y)y' + b(x, y)]$, where $a_x = b_y,\ a_y = -b_x$. An equation of this type is reduced by a conformal transformation to the form $y'' = 0$.

An analogous result for the case of the general group of analytic transformations was obtained by Tresse ([5]).

§ 4. **The space of linear elements of Euclidean connection, adjoined to the equation** $y'' = f(x, y, y')$. We shall assume that $m \neq 0$. Then, with the aid of ω^2, we reduce m to unity: $m = 1$. The last equation (8) gives $\omega^2 = p\omega^1 + q\tilde{\omega}^1 + r\tilde{\omega}^2$. The quantities p, q, r are invariants. From (4) we obtain

$$p = \frac{1}{(1 + {y'}^2)\hat{f}^2_{y'y'}} \times$$

$$\times\{-\hat{f}_{xy'} - y'\hat{f}_{yy'} - (1 + {y'}^2)\hat{f}_y\hat{f}_{y'y'} - 2y'\hat{f}\hat{f}_{y'y'} - (1 + {y'}^2)\hat{f}\hat{f}_{y'y'y'}\},$$

$$q = \frac{1}{(1 + {y'}^2)\hat{f}^2_{y'y'}} \times$$

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$$\times\{y'\hat f_{xy'y'}-\hat f_{yy'y'}+\hat f\hat f_{y'y'}-((1+{y'}^2)\hat f_y-y'\hat f)(3y'\hat f_{y'y'}+(1+{y'}^2)\hat f_{y'y'y'})\},\quad (12)$$

$$r=-3y'-(1+{y'}^2)\frac{\hat f_{y'y'y'}}{\hat f_{y'y'}}.$$

Substituting $\omega^2=p\omega^1+q\tilde\omega^1+r\tilde\omega^2$ into (1) and making the replacement $\omega^1=\theta^1,\ \tilde\omega^1=\theta^2,\ \tilde\omega^2+\omega^1=\theta^1_2=-\theta^2_1$, we obtain:

$$D\theta^1=[\theta^2\theta^1_2]+(q-1)[\theta^1\theta^2]+r[\theta^1\theta^2_1],\quad D\theta^2=[\theta^1\theta^2_1]+(r-p)[\theta^1\theta^2]+r[\theta^2\theta^2_1],$$

$$D\theta^2_1=(q-k)[\theta^1\theta^2]+r[\theta^1\theta^2_1].\quad (13)$$

These are the structure equations of the invariantly adjoining two-dimensional manifold of linear elements of Euclidean connection.

Consider the case $r=0$. The last equality (12) shows that the equation assumes the form

$$y''=(1+{y'}^2)\,[a(x,y)y'+b(x,y)+c(x,y)\sqrt{1+{y'}^2}].\quad (14)$$

Suppose now that $r=0,\ p=0$. Differentiating (13) exteriorly, we obtain $q-1=0$. Similarly, if $r=0,\ q-1=0$, then $p=0$. Thus, let $r=p=q-1=0$. Then we have a two-dimensional Riemannian space. Our equation in this case assumes the form (14), where $a(x,y)=-\partial\ln c(x,y)/\partial x,\ b(x,y)=\partial\ln c(x,y)/\partial y$. If $k=1$, then the space is flat and $\partial^2\ln c(x,y)/\partial x^2+\partial^2\ln c(x,y)/\partial y^2=0$. The types of equations obtained are invariant.

It can be shown that $y''=f(x,y,y')$ **is an equation of lines of unit curvature in the adjoined space of linear elements of Euclidean connection.**

In the case $r=p=q-1=k-1=0$ the space is flat; hence:

Theorem 3. In order that a two-parameter family of lines in the plane be carried by a conformal transformation into a family of circles–

of unit radius, it is necessary and sufficient that this family of curves be given by an equation of the form $y''=(1+{y'}^2)[a(x,y)y'+b(x,y)+c(x,y)\sqrt{1+{y'}^2}]$, where $a(x,y)=-\partial\ln c(x,y)/\partial x,\ b(x,y)=\partial\ln c(x,y)/\partial y,\ \partial^2\ln c(x,y)/\partial x^2+\partial^2\ln c(x,y)/\partial y^2=0$. An equation of this type is brought by a conformal transformation to the form $y''=-(1+{y'}^2)^{3/2}$.

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We shall call a line whose curvature is equal to zero a geodesic line. The equation of a geodesic is $\theta^2 = 0$, $\theta_1^2 = 0$, or, by virtue of (4),

$$y'' = f(x,y,y') - \hat{f}_{y'y'}(1+{y'}^2). \tag{15}$$

Equation (15) is invariantly associated with the equation $y'' = f(x,y,y')$. We shall call it the associated equation. We now require that equation (15) have the form (11) or (14). Then the equation $y'' = f(x,y,y')$ assumes, respectively, the form

$$y'' = (1+{y'}^2)\left[a(x,y)y' + b(x,y) + c(x,y)\sqrt{1+{y'}^2}\,\operatorname{arc\,tg} y' + d(x,y)\sqrt{1+{y'}^2}\right] \tag{16}$$

or

$$y'' = (1+{y'}^2)\left[a(x,y)y' + b(x,y) + c(x,y)\sqrt{1+{y'}^2}\,(\operatorname{arc\,tg} y')^2 + d(x,y)\sqrt{1+{y'}^2}\,\operatorname{arc\,tg} y' + e(x,y)\sqrt{1+{y'}^2}\right]. \tag{17}$$

The invariance of equations (14), (16), (17) is a special case of the following assertion:

Theorem 4. An equation of the form

$$y'' = (1+{y'}^2)[a(x,y)y' + b(x,y)] + (1+{y'}^2)^{3/2}R_n(\operatorname{arc\,tg} y'),$$

where R_n is a polynomial of degree n in $\operatorname{arc\,tg} y'$ with coefficients depending on (x,y), is transformed under an arbitrary conformal transformation into an equation of the same type with the same n.

For $n = 0, 1, 2$ we obtain, respectively, (14), (16), (17). The proof follows from the fact that an equation for which the associated equation has the form

$$y'' = (1+{y'}^2)(ay'+b) + (1+{y'}^2)^{3/2}\widetilde{R}_n(\operatorname{arc\,tg} y')$$

is written in the form

$$y'' = (1+{y'}^2)(\tilde{a}y'+\tilde{b}) + (1+{y'}^2)^{3/2}R_{n+2}(\operatorname{arc\,tg} y').$$

Finding the variation of the integral $\delta\int ds$, we obtain the extremality conditions for the curve $\theta^2 = 0$, $\theta_1^2 = \lambda\theta^1$:

$$\lambda(1+r^2-r_{12}) + (q-1-r^2+rp-r_1) = 0,$$

where r_1, r_{12} are the coefficients in the expansion $dr = r_1\theta^1 + r_2\theta^2 + r_{12}\theta_1^2$. If $1+r^2-r_{12} = 0$, then, by virtue of (4), (12), $y'' = f(x,y,y')$ assumes the form

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$$y'' = (1+y'^2)[a(x,y)y' + b(x,y) + c(x,y)y' \operatorname{arc\,tg} y' + d(x,y) \operatorname{arc\,tg} y']. \quad (18)$$

The condition $q - 1 - r^2 + rp - r_1 = 0$ gives $c_x(x,y) + d_y(x,y) = 0$.

The invariance of equations (11) and (18) is a special case of the following assertion:

Theorem 5. An equation of the form

$$y'' = (1+y'^2)[P_n(\operatorname{arctg} y') + y' Q_n(\operatorname{arctg} y')],$$

where R_n, Q_n are polynomials of degree n in $\operatorname{arc\,tg} y'$ with coefficients depending on (x,y), is transformed under an arbitrary conformal transformation into an equation of the same type with the same n.

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Note: Figure translations are in progress. See original paper for figures.

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