Reports of the Academy of Sciences of the USSR

1961

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1961. Volume 140, No. 1

MATHEMATICS

M. K. KERIMOV

ON THE THEORY OF THE SECOND VARIATION FOR DISCONTINUOUS VARIATIONAL PROBLEMS IN SPACE

(Presented by Academician A. A. Dorodnitsyn on 6 III 1961)

The present note is devoted to the theory of the second variation for discontinuous variational problems with movable ends in a space of many dimensions ([1]). We have chosen the nonparametric case, since in this case the following essential difference between parametric and nonparametric problems becomes apparent: in the parametric case ([2]), both primary and secondary extremals are polygonal lines, whereas in the nonparametric case the primary extremals are polygonal, while the secondary extremals have discontinuities of the first kind. Thus there arise discontinuous variational problems with discontinuous extremals. As in ([1]), we shall confine ourselves to the case in which the integrand in the integral J, for which the minimum is sought, has only one surface of discontinuity. However, our results are without difficulty carried over to the case of any finite number of discontinuity surfaces.

1. Let E_{102} be a polygonal extremal realizing the minimum of the functional J. To compute the second variation of the discontinuous functional J, we generalize the method proposed by Morse ([3]) for continuous variational problems. Let $\alpha_h(a), \beta_h(a), \gamma_h(a)$ $(h = 1, 2, \dots, n)$ be functions of class $C^{(2)}$ for a near $a = 0$. Substituting them into the primary end conditions ([1]) and differentiating with respect to a, at $a = 0$ we find

$$\eta_i^1 = \eta_i(x^1) = C_{ih}^1 \tau_h^1, \qquad \eta_i^2 = \eta_i(x^2) = C_{ih}^2 \tau_h^2,$$

$$\eta_i^{0-} = \eta_i^{0-}(x^0) = C_{ih}^{0-} \tau_h^0, \qquad \eta_i^{0+} = \eta_i^{0+}(x^0) = C_{ih}^{0+} \tau_h^0 \qquad (h,\ i = 1, 2, \dots, n), \tag{A}$$

where

$$\eta_i(x) = y_a(x, 0), \qquad \tau_h^1 = \alpha_h'(0), \qquad \tau_h^2 = \gamma_h'(0), \qquad \tau_h^0 = \beta_h'(0),$$

$$C^k_{ih}(0) = y^k_{ih}(0) - y_{ix}(x^k)x^k_h(0),$$

$$C^{0-}_{ih} = y^0_{ih}(0) - y^-_{ix}(x^0)x^0_h(0), \qquad C^{0+}_{ih} = y^0_{ih}(0) - y^+_{ix}(x^0)x^0_h(0)$$

$$(i, h = 1, 2, \dots, n;\ k = 1 \text{ or } 2).$$

We shall call conditions (A) the **secondary end conditions**. One can construct ([1]) a one-parameter family of admissible curves

$$y_i = y_i(x, a) \begin{cases} y^-_i(x, a), & x^1(a) \le x \le x^0(a), \\ y^+_i(x, a), & x^0(a) \le x \le x^2(a), \end{cases} \tag{1}$$

containing the given polygonal extremal E_{102}. The functions $\eta_i(x)$ and the constants $\tau^1_h, \tau^0_h, \tau^2_h$ will be called the **variations of the family** (1) along E_{102}.

Differentiating the function $J(a)$ twice with respect to a and putting $a = 0$, taking into account the primary transversality and discontinuity conditions, we obtain the expression

$$J''(0) = J_2(\eta, \tau) = b^1_{hl}\tau^1_h\tau^1_l + \left(b^{0-}_{hl} + b^{0+}_{hl}\right)\tau^0_h\tau^0_l + b^2_{hl}\tau^2_h\tau^2_l \tag{2}$$

$$+ \int_{x^1}^{x^0} 2\omega^1(x, \eta, \eta')\,dx + \int_{x^0}^{x^2} 2\omega^2(x, \eta, \eta')\,dx,$$

which we shall call the second variation of the functional J along E_{102}. In formula (2) b^1_{hl}, b^{0-}_{hl}, b^{0+}_{hl}, b^2_{hl} are rather complicated expressions depending on F^1, F^2 and their derivatives at the points $1, 0, 2$, and

$$2\omega^k(x, \eta, \eta') = F^k_{y_iy_j}\eta_i\eta_j + 2F^k_{y_iy'_j}\eta_i\eta'_j + F^k_{y'_iy'_j}\eta'_i\eta'_j$$

$$(i, j = 1, 2, \dots, n;\ k = 1 \text{ or } 2).$$

By an admissible set of variations $\eta_i(x)$, τ^1_h, τ^0_h, τ^2_h $(x^1 \le x \le x^2)$ we shall mean a set for which $\tau^1_h, \tau^0_h, \tau^2_h$ are constants, and the $\eta_i(x)$ belong to the class $D^{(1)}$ on $x^1 \le x \le x^2$, with the exception of the point $x = x^0$, where they have a discontinuity of the first kind subject to the condition

$$\eta^{0-}_i = C^{0-}_{ih}\tau^0_h, \qquad \eta^{0+}_i = C^{0+}_{ih}\tau^0_h. \tag{3}$$

2. We shall say that the broken extremal E_{102}, intersecting the manifolds M^1, M^2 at the points $1, 2$ with observance of the primary transversality conditions, satisfies the necessary condition of Jacobi in terms of the sign of the second variation (condition IV), if $J_2(\eta, \tau)$ along E_{102} is nonnegative for every admissible set of variations satisfying conditions (A).

Theorem 1. *In order that the broken extremal E_{102} realize a minimum of the functional J, it is necessary that it satisfy condition IV.*

The proof is based on the following lemma.

Lemma. *For every admissible set of variations $\eta_i(x)$, τ_h^1, τ_h^0, τ_h^2 along E_{102}, satisfying conditions (A), there exists a one-parameter family of admissible curves (1), containing E_{102} for $a = 0$, such that $\eta_i(x)$, τ_h^1, τ_h^0, τ_h^2 is the set corresponding to this family along E_{102}; the functions $y_i^-(x, a)$, $y_i^+(x, a)$ possess all the continuity and differentiability properties required in computing the second variation of the functional J along E_{102}.*

3. Let us formulate a new (so-called secondary) variational problem for the functional $J_2(\eta, \tau)$ in the class of admissible sets of variations $\eta_i(x)$, $\tau_h^1, \tau_h^0, \tau_h^2$, satisfying conditions (A). We have a problem of the same type as the primary variational problem (1), with the sole difference that now the functions $\eta_i(x)$ of the class $D^{(1)}$, having at $x = x^0$ a discontinuity of the first kind, are taken as admissible functions, while in place of the manifolds M^1, M^0 and M^2 there appear n-dimensional linear manifolds N^1, N^0, N^2 with equations (A).

A set $\eta_i(x)$, $\tau_h^1, \tau_h^0, \tau_h^2$, realizing a minimum of the functional $J_2(\eta, \tau)$, must satisfy the system of secondary Euler equations

$$\mathcal{J}_i(\eta) = \frac{d}{dx}\omega_{\eta_i'} - \omega_{\eta_i} \begin{cases} \dfrac{d}{dx}\omega^1_{\eta_i'} - \omega^1_{\eta_i} = 0, & x^1 \le x \le x^0, \\ \dfrac{d}{dx}\omega^2_{\eta_i'} - \omega^2_{\eta_i} = 0, & x^0 \le x \le x^2, \end{cases} \tag{4, 5}$$

$$(i = 1, 2, \dots, n)$$

and the boundary conditions

$$C^1_{ih}\xi^1_i(x^1) = b^1_{hl}\tau^1_l, \qquad C^2_{ih}\xi^2_i(x^2) = -\,b^2_{hl}\tau^2_l, \tag{6}$$

$$C^{0-}_{ih}\xi^1_i(x^0) - C^{0+}_{ih}\xi^2_i(x^0) = -\,(b^{0-}_{hl} + b^{0+}_{hl})\tau^0_l \quad (i, l, h = 1, 2, \dots, n), \tag{7}$$

where the notation has been introduced

$$\xi^1_i(x) = \omega^1_{\eta_i'}\,[x, \eta(x), \eta'(x)], \qquad \xi^2_i(x) = \omega^2_{\eta_i'}\,[x, \eta(x), \eta'(x)]. \tag{8}$$

We shall call (6) the secondary transversality conditions, and (7) the secondary corner condition.

A broken curve

$$\eta_i = \eta_i(x) \begin{cases} \eta_i^-(x), & x^1 \leqslant x \leqslant x^0, \\ \eta_i^+(x), & x^0 \leqslant x \leqslant x^2 \end{cases} \qquad (i = 1, 2, \dots, n),$$

whose arcs $\eta_i^-(x)$ and $\eta_i^+(x)$ satisfy equations (4), (5), respectively, on $x^1 \leqslant x \leqslant x^0$ and $x^0 \leqslant x \leqslant x^2$, and condition (7) for $x = x^0$, is called a secondary broken (or simply broken) extremal. Since at the points 1, 0, 2 of the polygonal extremal E_{102} the transversality condition is satisfied, it follows (cf. (3)) that in neighborhoods of these points the linear manifolds N^1, N^0, N^2 will be regular.

Theorem 2. *For any extremal $\eta_i^-(x)$, τ_h^1 $(x^1 \leqslant x \leqslant x^0)$ of equation (4), satisfying the first of conditions in (A) and (6) and intersecting the manifold N^0, there exists a unique extremal $\eta_i^+(x)$, τ_h^0 $(x^0 \leqslant x \leqslant x^2)$ of equation (5), satisfying together with $\eta_i^-(x)$ conditions (3) and (7).*

This theorem makes it possible, for each family of extremals of equation (4), to construct a unique supplementary family, which together with the first constitutes a family of broken extremals of the secondary variational problem.

Introduce canonical variables η_i, ξ_i^k by means of the equations

$$\xi_i^k = \omega^k_{\eta_i'}(x, \eta, \eta'), \tag{9}$$

from which one can find

$$\eta_i' = \Pi_i^k(x, \eta, \xi^k), \qquad i = 1, 2, \dots, n \tag{10}$$

($k = 1$ corresponds to the interval $x^1 \leqslant x \leqslant x^0$, and $k = 2$ to the interval $x^0 \leqslant x \leqslant x^2$). Write the Hamiltonian functions

$$\mathfrak{H}^k(x, \eta, \xi^k) = [\eta_i' \omega^k_{\eta_i'} - \omega^k]_{\eta_i' = \Pi_i^k} = \Pi_i^k \xi_i^k - \omega^k(x, \eta, \Pi^k) \tag{11}$$

($k = 1$ on $x^1 \leqslant x \leqslant x^0$ and $k = 2$ on $x^0 \leqslant x \leqslant x^2$).

The canonical systems of equations, equivalent to the Euler equations (5), (6), have the form

$$d\eta_i/dx = \mathfrak{H}^k_{\xi_i^k}, \qquad d\xi_i^k/dx = -\mathfrak{H}^k_{\eta_i}, \qquad i = 1, 2, \dots, n \tag{12}$$

($k = 1$ on $x^1 \leqslant x \leqslant x^0$ and $k = 2$ on $x^0 \leqslant x \leqslant x^2$).

Take system (12) for $k = 1$ with the initial conditions

$$\eta_i(x^1) = C_{il}\tau_l^1, \qquad \xi_i^1(x^1)C_{ih}^1 = b_{hl}^1\tau_l^1 \quad (i, h, l = 1, 2, \dots, n). \tag{13}$$

The system of $2n$ linear equations (13) has a maximal system of n linearly independent solutions $\eta_{il}(x^1)$, $\xi_{il}^1(x^1)$, τ_{hl}^1 $(l = 1, 2, \dots, n)$, since the rank of the matrix $\|C_{ih}\|$ is equal to n. Consequently, system (12) for $k = 1$ has n linearly independent solutions $\eta_{il}(x)$, $\xi_{il}^1(x)$ $(x^1 \leqslant x \leqslant x^0,\ l = 1, 2, \dots, n)$, taking the initial values $\eta_{il}(x^1)$, $\xi_{il}^1(x^1)$ $(l = 1, 2, \dots, n)$.

Theorem 3. *For any linearly independent system of solutions η_{il}, ξ_{il}^1, τ_{hl}^1 of equations (12) for $k = 1$, satisfying the initial conditions (13), there exists a unique supplementary system of linearly independent solutions η_{il}, ξ_{il}^2, τ_{hl}^0 of equations (12) for $k = 2$, and the resulting system of broken solutions of equations (12) is linearly independent.*

In what follows we shall denote this system simply by $\eta_{il}(x)$, $\xi_{il}(x)$, τ_{hl}.

Let two systems of variables x, η_i, η_i' and x, u_i, u_i' be given. Introduce the notation

$$\bar{\xi}_i^k(x) = \omega_{\eta_i'}^k(x, u, u'), \qquad i = 1, 2, \dots, n; \quad k = 1 \text{ or } 2.$$

Then

$$\eta_i\omega_{\eta_i}^k(x, u, u') + \eta_i'\bar{\xi}_i^k(x) = u_i\omega_{\eta_i}^k(x, \eta, \eta') + u_i'\xi_i^k(x),$$

$$\eta_i\mathcal{J}_i^k(u) - u_i\mathcal{J}_i^k(\eta) = \frac{d}{dx}Q^k(\eta; u),$$

where

$$Q^k(\eta; u) = \eta_i\bar{\xi}_i^k - u_i\xi_i^k, \quad k = 1 \text{ or } 2.$$

If η_i and u_i are a pair of discontinuous solutions of the system (4)–(5), then $Q^1(\eta; u) = \text{const}$ or $Q^2(\eta; u) = \text{const}$. If these constants are equal to zero, then $\eta_i(x)$, $u_i(x)$ are called mutually adjoint.

Theorem 4. *Any two solutions η_i, ξ_i, τ_h and $\bar{\eta}_i, \bar{\xi}_i, \bar{\tau}_h$ belonging to a system of discontinuous linearly independent solutions of equations (12) are mutually adjoint.*

A system of pairwise mutually adjoint discontinuous linearly independent solutions $\eta_{il}, \xi_{il}, \tau_{hl}$ will be called an **adjoint basis**, and the determinant

$$\Delta(x) \begin{cases} \Delta^-(x) = |\eta^-_{il}(x)|, & x^1 \le x \le x^0, \\ \Delta^+(x) = |\eta^+_{il}(x)|, & x^0 \le x \le x^2 \quad (i, l = 1, 2, \dots, n) \end{cases} \tag{14}$$

the **basis determinant**.

Every solution $\eta_i(x), \xi_i(x), \tau_h$ of problem (12)–(13) is linearly expressible in terms of the solutions $\eta_{il}, \xi_{il}, \tau_{hl}$ of the adjoint basis.

The totality of all solutions of the canonical system depending on an adjoint basis will be called an **adjoint family**.

Definition of a focal point. Corresponding to the value $x = x^3$ ($x^3 \ne x^1$, $x^3 \ne x^0$), a point of a nonsingular broken extremal E_{102}, transversal to the manifold M^1 at point 1, is called a **focal point of the manifold** M^1 if problem (4)–(5)–(13) has a discontinuous solution $u_i(x)$ such that $u_i(x^3) = 0$, but not all u_i are identically equal to zero on the interval $x^1 < x < x^3$. Focal points of the manifold M^2 are defined analogously.

Theorem 5. *The focal points of the manifold M^1 on E_{102} correspond to the zeros $x = x^3$, $x^3 \ne x^1$, $x^3 \ne x^0$, of the basis determinant (14).*

In (1), the Jacobi condition was proved in terms of focal points under certain restrictions connected with the application of the envelope theorem. Now we can prove this condition without the indicated restrictions, on the basis of the above theory of the second variation. Namely, one can prove:

Necessary Jacobi condition in terms of focal points (condition IV). *In order that a nonsingular broken extremal E_{102}, with the single corner point 0, transversal to the manifolds M^1 and M^2, realize a minimum of the functional J, it is necessary that between points 1 and 2 on E_{102} there be no focal points of the manifolds M^1 and M^2.*

Computing Center
Academy of Sciences of the USSR

Received
3 III 1961

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Note: Figure translations are in progress. See original paper for figures.