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Abstract

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ON THE SUPERCONVERGENCE OF SEQUENCES OF RATIONAL FUNCTIONS

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1. Let E be an arbitrary continuum in the plane P of the complex variable z; let $f(z)$ be a function continuous on E and analytic on the set of interior points of E; and let $r_n(z)$, $n = 1, 2, ...$, be a sequence of rational functions of the form

$$r_n(z) = \frac{a_{n0}z^n + a_{n1}z^{n-1} + \cdots + a_{nn}}{(z-\alpha_{n1})(z-\alpha_{n2})\cdots(z-\alpha_{nn})}, \qquad \alpha_{nk} \notin E \tag{1}$$

(if $\alpha_{nk} = \infty$, then the corresponding factor in the denominator of $r_n(z)$ is replaced by unity), converging uniformly on the continuum E to the function $f(z)$.

To condition (1) there belongs a series of results establishing a dependence between the rate of convergence on E of the sequence of rational functions $r_n(z)$, the poles $\{\alpha_{nk}\}$ of which have no limit points in a certain neighborhood of the continuum E, and the domain in which superconvergence of the sequence $r_n(z)$ takes place and to which, consequently, the function $f(z)$ is analytically continuable. In particular, if G_0 is a fixed domain containing E, the sequence $\{\alpha_{nk}\}$, $n = 1, 2, \ldots;\ k = 1, 2, \ldots, n$, has no limit points inside G_0, and the condition

$$\lim_{n\to\infty} \left[\max_{z\in E} |f(z) - r_n(z)|\right]^{1/n} = 0 \tag{2}$$

is satisfied, then the sequence of rational functions $r_n(z)$ of the form (1) converges uniformly inside G_0, and, consequently, the function $f(z)$ admits analytic continuation to the domain G_0; moreover, condition (2) remains valid on every closed set belonging to G_0. If condition (2) is replaced by the condition

$$\lim_{n\to\infty} \left[\max_{z\in E} |f(z) - r_n(z)|\right]^{1/n} = q < 1, \tag{3}$$

then convergence of the sequence $r_n(z)$ can be guaranteed only in a subdomain $g_0 \subset G_0$, depending on q and on the geometric properties of E and G_0.

On the other hand, Borel[2], studying functions of the form $\sum \frac{A_n}{z-z_n}$, raised the question of quasianalytic continuation of analytic functions through a cut; he obtained, in particular, the following result in this direction: if

$$f(z) = \sum \frac{A_n}{z - z_n}, \qquad \lim_{n\to\infty} \sqrt[n]{|A_n|} = 0,$$

and $f(z) = 0$ in some domain that contains no limit points of the sequence $\{z_n\}$, then $f(z) = 0$ at every point of the plane that is not a limit point of the sequence $\{z_n\}$ (and is not equal to z_n).

In the present paper we give some results on the superconvergence of sequences of rational functions of general form (1) in domains not intersecting the continuum E, and also on the [[unclear: text continues on next page]]

sequences by a quasianalytic continuation of functions defined on E, into a domain not intersecting E.

2. Denote by P^α the set of limit points of the sequence $\{\alpha_{nk}\}$, $n = 1, 2, \ldots;\ k = 1, 2, \ldots, n$, of the poles of the rational functions $r_n(z)$ of the form (1); $G^\alpha = P \setminus P^\alpha$ is an open set; $G^\alpha = \bigcup_k G_k^\alpha$, where G_k^α, $k = (0), 1, 2, \ldots$, are the connected components of the set G^α (we assume that there exists at least one domain G_k^α, $k \neq 0$, not intersecting the continuum E).

Theorem 1. Let the function $f(z)$ be defined only on the continuum E, and suppose there exists a sequence of rational functions $r_n(z)$ of the form (1) such that condition (2) is satisfied. Then: a) the sequence $r_n(z)$ converges on the set G^α uniformly on every closed set contained in G^α, and, consequently, the function

$$F(z) = \lim_{n\to\infty} r_n(z), \quad z \in G^\alpha,$$

is analytic on the set G^α; b) for every closed set e belonging to one of the domains G_k^α, we have

$$\lim_{n\to\infty} \left[\max_{z\in e} |F(z) - r_n(z)|\right]^{1/n} = 0;$$

c) the limiting function $F(z)$ is a quasianalytic continuation of the function $f(z)$ in the sense that the values of the function $F(z)$ on the set G^α are uniquely determined by the values of $f(z)$ on E (and hence also by the values of $F(z)$ in one of the domains G_k^α). More precisely, if $g_n(z)$ is a sequence of rational functions of the form

$$g_n(z) = \frac{b_{n0} z^n + b_{n1} z^{n-1} + \cdots + b_{nn}}{(z - \beta_{n1})(z - \beta_{n2}) \cdots (z - \beta_{nn})}, \quad \beta_{nk} \bar{\in} E,$$

different from $r_n(z)$ and such that

$$\lim_{n\to\infty} \left[\max_{z\in E} |f(z) - g_n(z)|\right]^{1/n} = 0,$$

then

$$F(z) = G(z) = \lim_{n\to\infty} g_n(z)$$

at every point of the set $G^\alpha \cap G^\beta$; d) if there exists an entire function $F_1(z)$ such that $F_1(z) = f(z)$, $z \in E$, then $F_1(z) = F(z)$, $z \in G^\alpha$; in particular, if $f(z) = 0$, $z \in E$, then $F(z) = 0$, $z \in G^\alpha$.

Assertion c) of Theorem 1 on quasianalytic continuation can be strengthened as follows: if the sequence $r_n(z)$ converges to the function $f(z)$ on a set E of positive capacity in such a way that condition (2) is satisfied, then the function $f(z)$ uniquely determines the limiting function

$$F(z) = \lim_{n\to\infty} r_n(z)$$

on any continuum on which the sequence $r_n(z)$ converges uniformly to $F(z)$ (all points of this continuum may be limit points of the sequence $\{\alpha_{nk}\}$). In terms of best approximations the corresponding result can be formulated as follows.

Theorem 2. Let $R_n(f;E)$ be the best approximation of the function $f(z)$ on the continuum E by rational functions $r_n(z)$ of the form (1). If

$$\lim_{n\to\infty} \sqrt[n]{R_n(f;E)} = 0 \tag{4}$$

and $f(z) = 0$ on a set $e \subset E$ of positive capacity, then $f(z) \equiv 0$ on the continuum E.

In Theorem 1 condition (2) obviously cannot be weakened. In this connection there exists a sequence $r_n(z)$ of the form (1), converging on E (to $f(z) \equiv 0$) with rate (2) and diverging at every point $z \bar{\in} G^\alpha$ (and $\bar{\in} E$). In Theorem 2 condition (4), apparently, may be replaced by the condition

$$\lim_{n\to\infty} \sqrt[n]{R_n(f;E)} = q < 1$$

(cf. Theorem 5 of (4)).

Theorem 1 is an analogue of the corresponding result of Walsh on analytic continuation and generalizes Borel' s result on quasianalytic continuation by means of functions of the form $\sum A_n/(z - z_n)$; let us note that Borel' s result can be strengthened: for the validity of the assertion stated above it is sufficient to require that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|A_n|} = q < 1.$$

3. Let G be an arbitrary domain not intersecting the continuum E; let $f(z)$ be a function defined only on E; if there exists a sequence of rational functions $r_n(z)$ of the form (1), $\{a_{nk}\}' \cap G = 0$, such that condition (2) is satisfied, then the sequence $r_n(z)$ converges also in the domain G to a function which is a quasianalytic continuation of the function $f(z)$ (Theorem 1). If condition (2) is replaced by condition (3), then superconvergence and

quasianalytic continuation can be guaranteed, generally speaking, only in some subdomain of the domain G. For illustration we give the simplest theorems for the case of circular domains; both the statements and the proofs of the theorems are analogous to the corresponding theorems of Walsh on analytic continuation [1] and admit analogous generalizations.

Theorem 3. Let a sequence of rational functions $r_n(z)$ of the form (1), whose poles satisfy the condition $1 < |a_{nk}| \leqslant R$, $n = N, N+1, \ldots$, converge to zero in the disk $|z| \leqslant 1$, and suppose that

$$\overline{\lim_{n\to\infty}} \left[\max_{|z|\leqslant 1} |r_n(z)|\right]^{1/n} = q < \frac{1}{R}. \tag{5}$$

Then the sequence $r_n(z)$ converges to zero for

$$|z| > \frac{R-q}{1-Rq}$$

uniformly for

$$|z| \geqslant Z > \frac{R-q}{1-Rq}.$$

For

$$|z| = \frac{R-q}{1-Rq}$$

there may be no convergence; indeed, the sequence

$$r_n(z) = \left(q\,\frac{1-Rz}{z-R}\right)^n, \qquad q < \frac{1}{R},$$

satisfies the conditions of the theorem and at the same time

$$r_n\left(\frac{R-q}{1-Rr}\right) = (-1)^n.$$

If convergence of the sequence $r_n(z)$ does take place for $|z| > R$, but condition (5) is violated, the limiting function for $|z| > R$ may be different from zero; indeed, the sequence

$$r_n(z) = \frac{z^n}{z^n - R^n}, \qquad R > 1,$$

converges to zero for $|z| \leqslant 1$ (and even for $|z| < R$) and to one for $|z| > R$, while in condition (5) $q = 1/R$.

The following theorems follow from Theorem 3.

Theorem 4. Let a sequence of rational functions $r_n(z)$ of the form (1), $1 < |a_{nk}| \leqslant R$, converge to a function $f(z)$ in the disk $|z| \leqslant 1$, and suppose that

$$\overline{\lim_{n\to\infty}} \left[\max_{|z|\leqslant 1} |f(z) - r_n(z)|\right]^{1/n} = q < \frac{1}{R^2}.$$

Then the sequence $r_n(z)$ converges for

$$|z| > \frac{R - q^{1/2}}{1 - Rq^{1/2}}$$

uniformly for

$$|z| \geqslant Z > \frac{R - q^{1/2}}{1 - Rq^{1/2}}$$

to an analytic function

$$F(z) = \lim_{n\to\infty} r_n(z), \qquad |z| > \frac{R - q^{1/2}}{1 - Rq^{1/2}}.$$

The function $F(z)$ is a quasianalytic continuation of the function $f(z)$ (i.e., the values of the function $F(z)$ are uniquely determined by the values of the function $f(z)$).

Theorem 5. Let a sequence of rational functions $r_n(z)$ of the form (1), where $a_{nk} = a_k$ does not depend on n, $1 < |a_k| \leqslant R$, converge to $f(z)$ in the disk $|z| \leqslant 1$, and suppose that

$$\overline{\lim_{n\to\infty}} \left[\max_{|z|\leqslant 1} |f(z) - r_n(z)|\right]^{1/n} = q < \frac{1}{R}.$$

Then the sequence $r_n(z)$ converges for

$$|z| > \frac{R - q}{1 - Rq}$$

uniformly for

$$|z| \geqslant Z > \frac{R - q}{1 - Rq}$$

to an analytic function $F(z)$, which is a quasianalytic continuation of the function $f(z)$.

Let K_1 and K_2 be arbitrary circular domains (i.e., domains bounded by a circle or a straight line); in view of the invariance of the problem under a fractional-linear transformation of the plane, the general case of approximation on K_1 by rational functions of the form (1) with the condition $a_{nk} \in K_2$ reduces to the canonical case considered above.

4. In the theorems stated above, the simplest geometric restrictions were imposed on the poles of the approximating rational functions. For sequences of rational functions $r_n(z)$ with a fixed set of poles $\{a_{nk}\}$, one can formulate stronger theorems on the set of convergence of the sequences $r_n(z)$ and on the quasi-analytic continuation of the function $f(z)$ determined by them: the set of convergence of a sequence $r_n(z)$, converging on E to $f(z)$ at the rate of a geometric progression, contains, generally speaking,

a certain set of the type $F_\sigma \supset E$ (possibly $= E$), whose complement is a certain "neighborhood" of the set of limit points of the sequence $\{a_{nk}\}$, and "inside" this set the convergence also has the order of a geometric progression.

The following assertion is an analogue of Theorem 3. Let E be a continuum (or, more generally, a set whose complement is regular in the sense of the Dirichlet problem); $G(z, \alpha)$ the Green's function for the complement of E with singularity at the point α; if a sequence of rational functions $r_n(z)$ of the form (1) converges on E to zero, and

$$\lim_{n\to\infty} \left[\max_{z\in E} |r_n(z)|\right]^{1/n} = q < 1,$$

then the sequence $r_n(z)$ converges to zero on the set of points for which

$$G(z) = \overline{\lim}_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} G(z, a_{nk}) \right) < \ln \frac{1}{q},$$

uniformly (and with the rate of a geometric progression) for

$$G(z) \le Q < \ln \frac{1}{q}$$

(the case $G(z) \equiv \infty,\ z \in E$, is possible). If $r_n(z)$ converges sufficiently rapidly on E, the full set of convergence of $r_n(z)$ is the whole plane except for an arbitrarily "rare" (in the sense of Hausdorff measure) set (cf. the monogenic functions of Borel (3); from the point of view of the properties of $f(z)$ on E, cf. (5,6)). We give the formulation of one possible theorem in this direction.

Theorem 6. *Let E be an arbitrary continuum; $f(z)$ a function defined only on E; if a sequence of rational functions $r_n(z)$ of the form (1) converges on E to $f(z)$ in such a way that*

$$\max_{z\in E} |f(z) - z_n(z)| \le \frac{C}{n^{\varepsilon n}}, \qquad \varepsilon > 0,$$

then for any lacunary sequence $n_1, n_2, ..., n_k, ...$ the sequence $r_{n_k}(z)$, $k = 1, 2, ...$, converges on a set $F \supset E$, whose complement has Hausdorff measure equal to zero of any finite order, to a function $F(z)$, which is a quasi-analytic continuation of the function $f(z)$.

We note that the set F can be represented as a sum of closed sets on each of which the order of approximation of the limiting function $F(z)$ by rational functions is also $n^{-\varepsilon_1 n}$, $\varepsilon_1 > 0$. These last remarks make it possible to understand the character of the quasi-analytic continuation determined by Theorems 1, 4, and 5.

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Note: Figure translations are in progress. See original paper for figures.