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Abstract

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ON THE CLASSIFICATION OF BOUNDED HOMOGENEOUS DOMAINS IN n-DIMENSIONAL COMPLEX SPACE

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A bounded domain in n-dimensional complex space is called **homogeneous** if, for any pair of its points, there exists an analytic automorphism of this domain taking one of the points to the other.

É. Cartan found all bounded homogeneous domains in complex spaces of dimensions 2 and 3 ([1]). It turned out that all these domains are symmetric domains. (A domain D is called symmetric if for each point $z_0 \in D$ there exists an analytic automorphism φ_0 of the domain D onto itself possessing the following properties: 1) $\varphi_0(z) = z$ only if $z = z_0$; 2) $\varphi_0^2(z) = z$ for all $z \in D$.) É. Cartan succeeded in finding all symmetric bounded homogeneous domains in n-dimensional complex space. He put forward the conjecture that every bounded homogeneous domain is automatically a symmetric domain.

A. Borel and J. Koszul proved that if a semisimple Lie group acts transitively in some bounded domain in a complex space, then this domain is a symmetric domain ([2,3]). Their result was strengthened by Hano ([4]). The first example of a bounded homogeneous nonsymmetric domain was constructed in ([5]).

The present work is devoted to the study of bounded homogeneous domains D possessing the following properties: α) D is homeomorphic to Euclidean space; β) there exists a one-to-one analytic mapping of the domain D onto some affinely homogeneous domain. The results obtained in the present work make it possible to classify completely all such domains, and also to find for them a canonical realization.

At first glance it seems that the restrictions imposed by us are too strong. However, they lead to a theory containing numerous examples. Moreover, there is every reason to expect that every bounded homogeneous domain possesses properties α) and β).

We shall agree to call domains possessing properties α) and β) **regular** domains.

It follows from our results that in 4-dimensional complex space there exists exactly one regular nonsymmetric domain. In 5-dimensional complex space there exist 6 irreducible regular domains, of which only 2 are symmetric. In 6-dimensional complex space the number of distinct regular domains is finite.

In 7-dimensional complex space the situation changes, namely, there exists a continuum of distinct, i.e. analytically inequivalent, regular domains.

More precisely, the following can be shown. Let $z_1, z_2, z_3, z_4, u_1, u_2, v$ be 7 complex variables. Introduce the following notation: $y_k = \operatorname{Im} z_k$,

$$k = 1, \dots, 4; \qquad F_1 = |u_1|^2 + |u_2|^2, \quad F_2 = |v|^2, \quad F_3 = \operatorname{Re} u_1\overline{v}, \quad F_4 =$$

$$= (\operatorname{Re} u_2\overline{v}) \cos\varphi + (\operatorname{Im} u_1\overline{v}) \sin\varphi.$$

Consider next the domain in 7-dimensional complex space defined by the inequalities

$$y_1 - F_1 > 0, \qquad (y_1 - F_1)(y_2 - F_2) - (y_3 - F_3)^2 - (y_4 - F_4)^2 > 0. \tag{1}$$

It can be proved that all domains of the indicated form are affinely homogeneous and analytically equivalent to bounded domains. Further, it can be shown that these domains are analytically equivalent to one another if and only if $\varphi_1 = \varphi_2$ (mod 2π), and that every regular domain in 7-dimensional space is analytically equivalent either to a domain of the form (1), or to one of a finite number of domains.

We now turn to the statement of the principal results of the present work and their method of proof.

It turns out that the classification of regular domains reduces to the classification of a certain class of Lie algebras, whose definition is given below.

A Lie algebra G is called normal if, for every element $g_0 \in G$, the characteristic values of the transformation $p_{g_0}(g) = [g_0, g]$ are real. It is not difficult to see that every normal Lie algebra is solvable.

A Lie algebra G is called a j-algebra if in G there exist an endomorphism j and a bilinear skew-symmetric form $\rho(x, y)$ for which the following relations hold:

$$j^2 = -1, \qquad \rho(j(x), j(y)) = \rho(x, y), \qquad \rho(j(x), x) > 0 \quad \text{for } x \neq 0,$$

$$[x, y] + j([j(x), y]) + j([x, j(y)]) - [j(x), j(y)] = 0;$$

$$\rho(x, y) = \omega([x, y]),$$

where ω is a certain linear form on G.

Theorem 1. *To every normal j-algebra G there corresponds a regular domain on which a group $\mathfrak{A}$ acts simply transitively and analytically, the Lie algebra of this group being isomorphic to G. Conversely, to every regular domain there corresponds a unique normal j-algebra.*

Thus the study and classification of regular domains reduces to the study of normal j-algebras. The proof of this theorem requires a detailed study of the structure of j-algebras.

We now give the simplest example of a normal j-algebra. Let G denote a $2m$-dimensional space over the field of real numbers, furnished with an endomorphism j and a skew-symmetric form $\rho(x, y)$, where j and $\rho(x, y)$ are subject to the following restrictions:

$$-j^2 = -1, \qquad \rho(j(x), j(y)) = \rho(x, y), \qquad \rho(j(x), x) > 0 \quad \text{for } x \neq 0.$$

Further, let R be some one-dimensional subspace of the space G. Put $G = R + j(R) + U$, where U is the orthogonal complement to $R + j(R)$, i.e. the set of all $x \in G$ such that $\rho(x, y) = 0$ for every $y \in R + j(R)$. Denote by r_0 that element of R for which $\rho(j(r_0), r_0) = 1$. Introduce in G the structure of a Lie algebra by setting

$$[r_0, u] = 0, \qquad [j(r_0), u] = \frac{1}{2}u$$

for every u,

$$[j(r_0), r_0] = r_0, \qquad [u, v] = \rho(u, v) r_0$$

for any $u, v \in U$. The j-algebra so constructed will be called elementary.

Let G be an arbitrary normal j-algebra. A decomposition of the algebra G into a sum of pairwise orthogonal subspaces

$$G = \sum_{k=1}^{p} G_k$$

is called a canonical decomposition if the following conditions are fulfilled: 1) G_k is a j-invariant subalgebra of the algebra G, isomorphic to an elementary j-algebra; 2) $G_k + G_s$, for $k < s$, is a j-invariant subalgebra of the algebra G, in which G_k is an ideal.

Theorem 2. *Every normal j-algebra possesses a canonical decomposition. The canonical decomposition is unique up to the order of numbering of the images* (8).

This theorem is the basis for the classification of normal j-algebras. Indeed, it follows from it that, in order to specify a normal j-algebra G, it is enough to specify p elementary j-algebras G_k ($k = 1, \ldots, p$) and, for each pair of integers k and s, to specify the operation of commutation between elements of G_k and of G_s. Obviously, specifying the operation of commutation between elements of G_k and G_s reduces to specifying a representation of the algebra G_s by derivations of the algebra G_k.

In other words, to each element $g \in G_s$ there is assigned a linear transformation $p_g^{k,s}$ of the space G_k, possessing the following properties:

$$p_g^{k,s}([x, y]) = [p_g^{k,s}(x), y] + [x, p_g^{k,s}(y)] \quad \text{for any } x, y \in G_k;$$

$$p_{[g_1, g_2]}^{k,s} = p_{g_1}^{k,s} p_{g_2}^{k,s} - p_{g_2}^{k,s} p_{g_1}^{k,s} \quad \text{for any } g_1, g_2 \in G_s. \tag{2}$$

Conversely, if for any k and s ($k < s$) to each element of G_s there is assigned a linear transformation $p_g^{k,s}$ of the space G_k, so that (2) holds, then one can naturally define a bilinear skew-symmetric mapping $G \times G$ into G, which we shall denote by $[x, y]$. In order that this mapping define in G the structure of a j-algebra, it is necessary and sufficient that the following identities hold:

$$[x, y] + j([j(x), y]) + j([x, j(y)]) - [j(x), j(y)] = 0; \tag{3}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \tag{4}$$

for all $x, y, z \in G$.

It is not difficult to see that identity (3) holds for any $x, y \in G_k$. Consequently, it remains to require that it hold for $x \in G_s$, $y \in G_k$ ($k < s$). Similarly, identity (6) holds for any x, y, z, provided it holds for $x \in G_s$, $y \in G_k$, $z \in G_t$ ($s > k > t$).

Thus, the description of arbitrary normal j-algebras reduces to the description of j-algebras of ranks 2 and 3. (The rank is the number of summands in the canonical decomposition.) The description of j-algebras of ranks 2 and 3 is given in the paper ([8]).

We now pass to the canonical realization of regular domains.

Theorem 3*. *Every regular domain is analytically equivalent to some affinely homogeneous Siegel domain of the second kind. Analytically equivalent affinely homogeneous Siegel domains of the second kind are affinely equivalent.*

Thus, it follows from Theorem 3 that every regular domain has, and essentially uniquely, a canonical realization in the form of some affinely homogeneous Siegel domain of the second kind (for the definition of Siegel domains of the second kind, see ([9])).

As is known, a Siegel domain of the second kind consists of points of the form (z, u) satisfying the following condition: $\operatorname{Im} z - F(u, u) \in V$, where V is a certain cone, and $F(u, u)$ is a vector Hermitian form. The transformation $z \to \lambda^2 z,\ u \to \lambda u$ is an analytic automorphism. Let g_0 be the directing vector of this one-parameter group; we may identify it with some element of the Lie algebra of the group of analytic automorphisms of our domain.

It turns out that the element g_0 admits the following description in terms of the internal properties of the Lie algebra G: 1) if $[g_0, x] = \lambda x$, then $[g_0, j(x)] = \mu x$, $\lambda + \mu = 1$; 2) the transformation $x \to [g_0, x]$ is semisimple and all its possible eigenvalues are $1, 0, \frac{1}{2}$. It turns out that the converse is also true, i.e. if in G there is an element g_0 possessing the properties indicated above,

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* S. G. Gindikin pointed out to me that in ([6]) there is an error, namely, the classification of the class of Siegel domains of the second kind indicated there is not complete. For pointing out this error, I express my gratitude to S. G. Gindikin. His remark had no effect on the proof of Theorem 3.

then to G there corresponds some domain of the second kind. The existence and uniqueness of the element g_0 are easily derived from Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.