SOLVABILITY OF ONE TYPE OF NONLINEAR BOUNDARY-VALUE PROBLEM

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1961

SovietRxiv

Abstract

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MATHEMATICS

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SOLVABILITY OF ONE TYPE OF NONLINEAR BOUNDARY-VALUE PROBLEM

(Presented by Academician P. Ya. Kochina on 10 III 1961)

We shall extend the results of ([5]) to the case of a more general problem (1).

1. Let L be a simple smooth closed contour in the plane of the complex variable, enclosing the origin of coordinates; let D^+ and D^- be, respectively, the interior and exterior domains into which L divides the plane; let $H(\alpha, N)$ be the set of functions defined on L and satisfying the Hölder condition:

$$|\varphi(t_1) - \varphi(t_2)| \le N|t_1 - t_2|^\alpha, \quad t_1, t_2 \in L, \quad 0 < \alpha \le 1;$$

let $W^+(W^-)$ be the space of functions analytic in $D^+(D^-)$ and continuous in $\overline{D^+}(\overline{D^-})$, with norm

$$\|\varphi^+(z)\| = \max_{z \in \overline{D^+}} |\varphi^+(z)| = \max_{t \in L} |\varphi^+(t)|;$$

respectively,

$$\|\varphi^-(z)\| = \max_{z \in \overline{D^-}} |\varphi^-(z)| = \max_{z \in L} |\varphi^-(z)|.$$

Statement of the problem: it is required to find two functions $\Phi^+(z) \in W^+$ and $\Phi^-(z) \in W^-$, whose boundary values satisfy on L the relation

$$[\Phi^+(t)]^n + h(t, \Phi^+(t)) = G(t)\Phi^-(t), \tag{1}$$

where $n \ge 2$ is an integer, $G(t) \in H(\alpha, N)$ and is everywhere different from zero; $h(t, u)$ is a function defined for $t \in L$ and $u = \varphi^+(z) \in W^+$.

We shall not consider the problem in such a general formulation, but shall require that $h(t, u)$ satisfy certain additional conditions, which we divide into two groups.

a)

$$|h(t, u)| \le M(1 + |u|^{n-\varepsilon}), \quad \varepsilon > 0;$$

b)

$$\begin{aligned} |h(t_1, u_1) - h(t_2, u_2)| \le M\big[(1 + \tilde{u}^{n-\varepsilon})|t_1 - t_2|^\alpha \\ +(1 + \tilde{u}^{n-1-\varepsilon})|u_1 - u_2|\big], \quad \tilde{u} = \max(|u_1|, |u_2|); \end{aligned} \tag{2}$$

c) $h(t,\varphi^+(z)) \in W^+$ with respect to z, t being a parameter;

d)
$$|h(t_1,u_1) - h(t_1,u_2) - h(t_2,u_1) + h(t_2,u_2)| \leq \\ \leq M(1+\tilde{u}^{\,n-1-\varepsilon})|t_1 - t_2|^{\alpha}|u_1 - u_2|. \tag{3}$$

An example of a function satisfying both conditions is

$$h(t,u) = \sum_1^{n-1} a_j(t)u^j, \quad a_j(t) \in H(\alpha, N).$$

The operator determined by the function $h(t,u)$ in W^+ will be denoted by $h\varphi^+$.

Theorem 1. If $h(t,u)$ satisfies conditions (2) and $\varkappa = \operatorname{ind} G(t) \geqslant 0$, then problem (1) is solvable.

Theorem 2. If $h(t,u)$ satisfies conditions (2) and (3) and $\varkappa \geqslant 0$, then the solution of problem (1) can be obtained by the method of successive approximations.

The proofs will be given below.

2. Repeating the transformations that lead to the solution of problem (1) in (5), we obtain:

$$\Phi^+(z) = e^{\frac{1}{n}\Gamma^+(z)}\left[P_\varkappa(z) - S^+h\Phi^+\right]^{1/n},$$

$$\Phi^-(z) = e^{\Gamma^-(z)}z^{-\varkappa}\left[P_\varkappa(z) - S^+h\Phi^+\right], \tag{4}$$

where

$$S\varphi = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}e^{-\Gamma^+(\tau)}\,d\tau, \qquad z \in D^+ \bigcup D^-.$$

The branch points of $\Phi^+(z)$ are excluded; therefore problem (4) in fact splits into n problems corresponding to the different branches of the radical. Problems (4) and (1) are equivalent; hence it is enough to prove the solvability of problem (4). For this purpose set $P_\varkappa(z) \equiv C$ and consider in W^+ the operator

$$A_0(C)\varphi^+ = e^{\frac{1}{n}\Gamma^+(z)}\{C - S^+h\varphi^+\}^{1/n},$$

where a definite branch of the radical is fixed. Denote by $S(R)$ the closed sphere in W^+ of radius R: $\|\varphi^+(z)\| \leqslant R$; $S(R,K)$ is the set of functions from $S(R)$ whose boundary values belong to $H(\alpha, K)$; $S(R,K)$ is a convex compact set.

Let us estimate $\|S^+h\varphi^+\|$:

a) $h(t,u)$ satisfies conditions (1) and $\varphi^+(z) \in S(R,K)$, $R \geqslant 1$.

$$\begin{aligned}
\|S^+h\varphi^+\| &= \max_{t\in L}\left|\frac{1}{2}h(t,\varphi^+(t))e^{-\Gamma^+(t)} + \frac{1}{2\pi i}\int_L \frac{h(\tau,\varphi^+(\tau))}{\tau - t}e^{-\Gamma^+(\tau)}\,d\tau\right| \\
&\leqslant \max_{t\in L}\left|h(t,\varphi^+(t))e^{-\Gamma^+(t)}\right| + \max_{t\in L}\left|\frac{1}{2\pi i}\int_L \frac{h(\tau,\varphi^+(\tau)) - h(t,\varphi^+(t))}{\tau - t}e^{-\Gamma^+(\tau)}\,d\tau\right| \\
&\leqslant M(1 + \|\varphi^+(z)\|^{n-\varepsilon})\|e^{-\Gamma^+(z)}\| \\
&\quad + M\|e^{-\Gamma^+(z)}\|\Big[(1+\|\varphi^+(z)\|^{n-\varepsilon}) + (1 + \|\varphi^+(z)\|^{n-1-\varepsilon})\Big]\frac{1}{2\pi}\int_L \frac{ds}{|\tau - t|^\alpha} \\
&\leqslant 2MR^{n-1-\varepsilon}\|e^{-\Gamma^+(z)}\|\left[R + (R+K)\frac{1}{2\pi}\int_L\frac{ds}{|\tau - t|^\alpha}\right].
\end{aligned}$$

b) $h(t,u)$ satisfies conditions (2) and (3), $\varphi^+(z) \in S(R)$, $R \geqslant 1$,

$$\begin{aligned}
\|S^+h\varphi^+\| &\leqslant \max_{t\in L}\left|h(t,\varphi^+(t))e^{-\Gamma^+(t)}\right| \\
&\quad + \max_{t\in L}\left|\frac{1}{2\pi i}\int_L \frac{h(\tau,\varphi^+(\tau)) - h(t,\varphi^+(\tau))}{\tau - t}e^{-\Gamma^+(\tau)}\,d\tau\right| \\
&\leqslant 2MR^{n-\varepsilon}\|e^{-\Gamma^+(z)}\|\left[1 + \frac{1}{2\pi}\int_L \frac{ds}{|\tau - t|^\alpha}\right].
\end{aligned}$$

Denote

$$b = \frac{1}{2\pi}\int_L \frac{ds}{|\tau - t|^\alpha}, \qquad \Gamma = \max(1, \|e^{\Gamma^+(z)}\|, \|e^{-\Gamma^+(z)}\|),$$

and let r and q be numbers $\geqslant 1$.

Lemma 1. If $h(t,u)$ satisfies conditions (1), $R = r\sqrt[n]{|C|}$, $K = q\sqrt[n]{|C|}$, $|C|^{\varepsilon/n} > 4M\Gamma r^{n-1}[r + (r+q)b]$, then the operator $A_0(C)$ is continuous on $S(R,K)$, and

$$A_0(C)S(R,K) \subset S\big[(2\Gamma|C|)^{1/n}\big].$$

Proof. Let $\varphi^+(z) \in S(R,K)$. From estimates a) it follows that

$$\|A_0(C)\varphi^+\|^{\Gamma 1/n}\big[|C| - \|S^+h\varphi^+\|\big]^{1/n} < [2\Gamma|C|]^{1/n},$$

i.e.

$$A_0(C)S(R,K) \subset S\big[(2\Gamma|C|)^{1/n}\big].$$

From the same estimates we obtain

$$|C| - \|S^+h\varphi^+\| \geq |C| - \frac{1}{2}|C| = \frac{1}{2}|C| > 0,$$

therefore

$$\operatorname{Re} e^{-i\arg C}\,[C - S^+ h\varphi^+] > 0.$$

Since the radical in the half-plane

$$\operatorname{Re} e^{-i\arg C} z > 0$$

is a continuous operator in W^+, it remains only to establish the continuity of S^+h on $S(R, K)$. The latter is a known fact ([4]). The lemma is proved.

Lemma 2. If $h(t, u)$ satisfies conditions (1),

$$R = r(|C|)^{1/n}, \qquad K = q(|C|)^{1/n},$$

$$|C|^{\varepsilon/n} > \max\left\{6M\Gamma r^{n-1}[r + (r+q)b],\ \frac{3}{2}(1+2p)Mr^{n-1}\left[r(\Gamma + pN) + q\Gamma\right]\right\},$$

then

$$A_0(C)S(R,K) \subset S\big[(2\Gamma|C|)^{1/n}\big], \quad (pN+1)(2\Gamma|C|)^{1/n}.$$

Proof. Since the hypotheses of the lemma ensure the validity of Lemma 1, it suffices to prove that the limiting values $A_0(C)\varphi^+$, where $\varphi^+(z) \in S(R,K)$, belong to

$$H\big[\alpha, (pN+1)(2\Gamma|C|)^{1/n}\big].$$

Let $\varphi^+(z) \in S(R,K)$. Then

$$|h(\tau_1, \varphi^+(\tau_1)) - h(\tau_2, \varphi^+(\tau_2))| \le M\big[(1 + R^{n-\varepsilon}) + (1 + R^{n-1-\varepsilon})K\big]|\tau_1 - \tau_2|^\alpha$$
$$\le 2MR^{n-1-\varepsilon}(R+K)|\tau_1 - \tau_2|^\alpha.$$

Since $G(t) \in H(\alpha, N)$, we have

$$e^{-\Gamma^+(t)} \quad \text{and} \quad e^{\frac{1}{n}\Gamma^+(t)} \in H(\alpha, pN),$$

therefore, by (3), the values of the transformed function $S^+h\varphi^+$ at the points L belong to

$$H(\alpha,\, p\,2MR^{n-1-\varepsilon}[(R+K)\Gamma + pNR]),$$

where p is some number independent of R and K. Further,

$$\big|[C - \tfrac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - S^+(t_1)h\varphi^+]^{1/n}$$
$$-[C - \tfrac{1}{2}e^{-\Gamma^+(t_2)}h(t_2)\varphi^+ - S^+(t_2)h\varphi^+]^{1/n}\big| =$$
$$\frac{\big|\frac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - \frac{1}{2}e^{-\Gamma^+(t_2)}h(t_2)\varphi^+ + S^+(t_1)h\varphi^+ - S^+(t_2)h\varphi^+\big|}{\big|\sum_1^n [C - \frac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - S^+(t_1)h\varphi^+]^{(n-k)/n}[C - \frac{1}{2}e^{-\Gamma^+(t_2)}h(t_2)\varphi^+ - S^+(t_2)h\varphi^+]^{(k-1)/n}\big|}.$$

Here $h(t)\varphi^+$ (as also $S^+(t)\alpha$) denotes the value of the transformed function at the point t. For the denominator we have the estimate

$$\left|\sum_1^n [C - \tfrac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - S^+(t_1)h\varphi^+]^{(n-k)/n}[C - \tfrac{1}{2}e^{-\Gamma^+(t_2)}h(t_2)\varphi^+\right.$$

$$-S^+(t_2)h\varphi^+]^{(k-1)/n}\big| \geq \left[|C| - \|S^+h\varphi^+\|\right]^{(n-1)/n} \left| 1 + \sum_1^{n-1} \left[1 + \frac{\frac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - \frac{1}{2}e^{-\Gamma^+(t_2)}h(t_2)\varphi^+ + S^+(t_1)h\varphi^+}{C - \frac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - S^+(t_1)h\varphi^+} \right. \right.$$

$$> \left[|C| - \|S^+h\varphi^+\|\right]^{(n-1)/n},$$

since the modulus of the second term in square brackets under the summation sign does not exceed

$$\frac{2\|S^+h\varphi^+\|}{|C| - \|S^+h\varphi^+\|} < 1.$$

Taking this inequality into account, we obtain

$$\big|[C - \tfrac{1}{2}e^{-\Gamma^+(t_1)}h(t_1)\varphi^+ - S^+(t_1)h\varphi^+]^{1/n} - [C - \tfrac{1}{2}e^{-\Gamma^+(t_2)}h(t_2)\varphi^+$$

$$-S^+(t_2)h\varphi^+]^{1/n}\big| \leq \frac{(\frac{1}{2}+p)\,2MR^{n-1-\varepsilon}[(R+K)\Gamma + pNR]}{(|C| - \|S^+h\varphi^+\|)^{(n-1)/n}}|t_1 - t_2|^\alpha$$

$$< \frac{(1+2p)Mr^{n-1}[(r+q)\Gamma + rpN]}{(2/3)^{1-1/n}|C|^{\varepsilon/n}}(|C|)^{1/n}|t_1 - t_2|^\alpha < (|C|)^{1/n}|t_1 - t_2|^\alpha.$$

Thus the lemma is proved, since

$$e^{\frac{1}{n}\Gamma^+(t)}\left[C - \frac{1}{2e^{-\Gamma^+(t)}}h(t)\varphi^+ - S^+(t)h\varphi^+\right]^{1/n} \in H\left[a, pN(2|C|)^{1/n}\right] + (\Gamma|C|)^{1/n} \subset$$

$$\subset H(\alpha, (2\Gamma|C|)^{1/n}(pN+1)).$$

Similarly, inequality b) permits one to prove the following assertions:

Lemma 3. If $h(t,u)$ satisfies conditions (2) and (3), $R = r(|C|)^{1/n}$ and $|C|^{\varepsilon/n} > 4M\Gamma r^n(1+b)$, then the operator $A_0(C)$ is continuous on $S(R)$ and

$$A_0(C)S(R) \subset S\left[(2\Gamma|C|)^{1/n}\right].$$

Lemma 4. If $h(t,u)$ satisfies conditions (2) and (3), $R = r(|C|)^{1/n}$ and $|C|^{\varepsilon/n} > 6\Gamma Mr(1+b)$, then $A_0(C)$ on $S(R)$ is a contraction operator.

3. Proof of Theorem 1. Let $|C|$ satisfy the conditions of Lemma 2, $r = (2\Gamma)^{1/n}$ and $q = (pN+1)(2\Gamma)^{1/n}$. Then $A_0(C)S(R,K) \subset S(R,K)$, i.e. the operator maps the set $S(R,K)$ into itself. Since $S(R,K)$ is a convex compact set and $A_0(C)$ is an operator continuous on $S(R,K)$, Schauder' s principle is applicable to $A_0(C)$. Consequently, in $S(R,K)$ the operator $A_0(C)$ has a fixed point $\varphi_0^+(z)$: $\varphi_0^+(z) = A_0(C)\varphi_0^+$. The theorem is proved.

Proof of Theorem 2. Let $|C|$ satisfy the conditions of Lemma 4 and $r = (2\Gamma)^{1/n}$. Then $A_0(C)S(R) \subset S(R)$, and on $S(R)$ the operator $A_0(C)$ will be a contraction operator. $S(R)$ is a closed set of the complete space W^+; therefore Banach' s principle is applicable to $A_0(C)$ on $S(R)$. The theorem is proved.

Remark. The solvability of problem (1) is ensured to a significant extent by the presence of the parameter C. Therefore the assumption $\varkappa \geq 0$ is essential in our reasoning. In the case $\varkappa < 0$, the question of solvability of problem (1), with a small exception, remains open. The exception is constituted by problem (1), considered in [5].

4. The problem

$$\Phi^+(t) = G(t)[\Phi^-(t)]^n + h(t, \Phi^-(t)) \tag{5}$$

by means of the change of variable $z = 1/w$ passes into the problem ($\tau = 1/t$)

$$\Phi_1^-(\tau) = G\left(\frac{1}{\tau}\right)[\Phi_1^+(\tau)]^n + h\left(\frac{1}{\tau}, \Phi_1^+(\tau)\right)$$

or

$$[\Phi_1^+(\tau)]^n + h_1(\tau, \Phi_1^+(\tau)) = G_1(\tau)\Phi_1^-(\tau), \tag{6}$$

where $h_1(\tau, \Phi_1^+(\tau)) = h(1/\tau, \Phi_1^+(\tau))/G(1/\tau)$. But problem (6) coincides with problem (1). Moreover,

$$\operatorname{ind} G_1(\tau) = \operatorname{ind} \frac{1}{G(1/\tau)} = \operatorname{ind} G(t) = \varkappa,$$

and $h_1(\tau, u)$ satisfies conditions (2) and (3), if $h(t, u)$ satisfies them. Therefore Theorems 1 and 2 also hold for problem (5).

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Received
23 II 1961

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.