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D. V. IONESCU

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Abstract

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MATHEMATICS

D. V. IONESCU

REPRESENTATION OF THE DIVIDED DIFFERENCE OF ORDER (m, n) OF A FUNCTION OF TWO VARIABLES BY A DOUBLE INTEGRAL. II

(Presented by Academician A. N. Kolmogorov on 12 V 1961)

1. In the preceding note ([1]) we proved that the divided difference of order (m, n) of a function $f(x, y)$ at the nodes (x_i, y_k), where $x_0 < x_1 < \cdots < x_m$, $y_0 < y_1 < \cdots < y_n$, can be represented by the formula

$$\left[\begin{array}{c} x_0, x_1, \dots, x_m; f \\ y_0, y_1, \dots, y_n \end{array}\right] = \iint_D \Phi(x, y) \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \, dx \, dy, \tag{1}$$

where D is the rectangle defined by the inequalities $x_0 \leq x \leq x_m$, $y_0 \leq y \leq y_n$, and the function $\Phi(x, y)$ coincides in each rectangle D_i^k, defined by the inequalities $x_i \leq x \leq x_{i+1}$, $y_k \leq y \leq y_{k+1}$, with the function $\varphi_i^k(x, y)$, defined by the formula

$$\varphi_i^k(x, y) = \frac{(-1)^{m-n}}{(m-1)!(n-1)!} \sum_{\alpha=0}^{i} \sum_{\beta=0}^{k} C_\alpha^\beta (x - x_\alpha)^{m-1} (y - y_\beta)^{n-1}; \tag{2}$$

the coefficients C_α^β are given by the formula

$$C_\alpha^\beta = (-1)^{\alpha+\beta} \frac{V(x_0, x_1, \dots, x_{\alpha-1}, x_{\alpha+1}, \dots, x_m)}{V(x_0, x_1, \dots, x_m)} \frac{V(y_0, y_1, \dots, y_{\beta-1}, y_{\beta+1}, \dots, y_n)}{V(y_0, y_1, \dots, y_n)}. \tag{3}$$

The expansions

$$\frac{1}{(x - x_0)(x - x_1) \cdots (x - x_m)} = \sum_{\alpha=0}^{m} \frac{A_\alpha}{x - x_\alpha},$$

$$\frac{1}{(y-y_0)(y-y_1)\cdots(y-y_n)} = \sum_{\beta=0}^{n} \frac{B_\beta}{y-y_\beta} \tag{4}$$

show that one may also write

$$C_\alpha^\beta = (-1)^{m+n} A_\alpha B_\beta. \tag{5}$$

In the present note we shall prove that the function $\Phi(x, y)$, which vanishes on the sides of the rectangle D, has constant sign inside this rectangle.

2. Consider the functions $\varphi_i^k(x, y)$ associated with the rectangles D_i^k lying between the lines $y = y_k$ and $y = y_{k+1}$, and study the functions of x, $\chi_i(x) = \varphi_i^k(x, y)$, where y has a fixed value from the interval (y_k, y_{k+1}), if $k = 0, 1, \dots, n-2$, or from the interval (y_{n-1}, y_n), if $k = n-1$. It is proved that the functions $\chi_i(x)$ satisfy the boundary conditions

$$\chi_0^{(j)}(x_0) = 0, \qquad \chi_i^{(j)}(x_i) = \chi_{i-1}^{(j)}(x_i), \qquad \chi_{m-1}^{(j)}(x_m) = 0 \tag{6}$$

for $i = 1, 2, \dots, m-1$ and $j = 0, 1, \dots, m-2$.

Similarly, as functions of y, $\theta_k(y) = \varphi_i^k(x, y)$, where x has a fixed value from the interval (x_i, x_{i+1}), if $i = 0, 1, \dots, m-2$, or from the interval (x_{m-1}, x_m), if $i = m-1$, satisfy the boundary conditions

$$\theta_0^{(j)}(y_0) = 0, \qquad \theta_k^{(j)}(y_k) = \theta_{k-1}^{(j)}(y_k), \qquad \theta_{n-1}^{(j)}(y_n) = 0 \tag{7}$$

for $k = 1, 2, \dots, n-1$ and $j = 0, 1, \dots, n-2$.

3. We shall assume that the derivative $\chi_i^{(m-1)}(x)$ does not vanish on the interval (x_0, x_m). This derivative is constant with respect to x, and we have

$$\chi_i^{(m-1)}(x) = \frac{A_0 + A_1 + \cdots + A_i}{(n-1)!} \left[B_0(y-y_0)^{n-1} + B_1(y-y_1)^{n-1} + \cdots \right.$$
$$\left. \cdots + B_k(y-y_k)^{n-1} \right]. \tag{8}$$

We proved in (2) that $A_0 + A_1 + \cdots + A_i \neq 0$, and it remains for us to prove that the sum

$$h_k = B_0(y-y_0)^{n-1} + B_1(y-y_1)^{n-1} + \cdots + B_k(y-y_k)^{n-1} \tag{9}$$

is not equal to zero for $k = 0, 1, \dots, n-1$, where y has a fixed value from the interval (y_k, y_{k+1}), if $k = 0, 1, \dots, n-2$, or from (y_{n-1}, y_n), if $k = n-1$. We assume that $n > 1$.

Considering the expansion

$$\frac{(y-Y)^{n-1}}{(Y-y_0)(Y-y_1)\cdots(Y-y_n)} = \sum_{j=0}^{n} \frac{B'_j}{Y-y_j}, \tag{10}$$

one may write

$$h_k = B'_0 + B'_1 + \cdots + B'_k. \tag{11}$$

Introducing the function

$$f(Y) = \frac{(y-Y)^{n-1}}{(Y-y_0)(Y-y_1)\cdots(Y-y_k)}, \tag{12}$$

we prove that

$$h_k = -[y_{k+1}, y_{k+2}, \dots, y_n; f], \tag{13}$$

and, by virtue of a known theorem, we have

$$h_k = -\frac{f^{(n-k-1)}(\bar{Y})}{(n-k-1)!}, \tag{14}$$

where $y_{k+1} < \bar{Y} < y_n$.

4. We have

$$f^{(n-k-1)}(Y) = (-1)^n (n-k-1)! \sum_{i=0}^{k} A_i \frac{(y_i-y)^{n-1}}{(y_i-Y)^{n-k}},$$

and, introducing a function of η

$$g(\eta) = \frac{(\eta-y)^{n-1}}{(\eta-Y)^{n-k}}, \tag{15}$$

we obtain

$$f^{(n-k-1)}(Y) = (-1)^n (n-k-1)! [y_0, y_1, \dots, y_k; g] =$$

$$= (-1)^n \frac{(n-k-1)!}{k!} g^{(k)}(\bar{\eta}), \tag{16}$$

where $y_0 < \bar{\eta} < y_k$.

It can be proved that

$$g^{(k)}(\eta) = (-1)^k \frac{(n-1)!}{(n-k-1)!} \frac{(Y-y)^k (\eta - y)^{n-k-1}}{(\eta - Y)^n}. \tag{17}$$

Let us return to formulas (14), (16), (17) and denote by $\bar{\eta}^*$ the number corresponding to $\bar{Y}$ in formula (16). We have

$$h_k = (-1)^{n+k+1} \binom{n-1}{k} \frac{(\bar{Y}-y)^k (\bar{\eta}^* - y)^{n-k-1}}{(\bar{\eta}^* - \bar{Y})^n}, \tag{18}$$

and this proves that $h_k \neq 0$, since

$$\bar{\eta}^* < y_k < y \leq y_{k+1} < \bar{Y}. \tag{19}$$

The preceding arguments were valid for $k = 1, 2, \dots, n-2$, but one can verify directly that we also have $h_0 \neq 0$ and $h_{n-1} \neq 0$.

5. It can be proved in the same way that the derivative $\theta_k^{(n-1)}(y)$ does not vanish on the interval (y_k, y_{k+1}). We shall restrict ourselves to considering the functions $\theta_k(y)$ corresponding to $i = 0$, i.e.

$$\theta_k(y) = \frac{(-1)^{m-n}}{(m-1)!(n-1)!} \left[C_0^0 (y-y_0)^{n-1} + C_0^1 (y-y_1)^{n-1} + \cdots \right.$$
$$\left. \cdots + C_0^k \times (y-y_k)^{n-1}\right] (x-x_0)^{m-1}. \tag{20}$$

For these functions we have

$$\theta_k^{(n-1)}(y) = \frac{(-1)^{m-n}}{(m-1)!} A_0 (B_0 + B_1 + \cdots + B_k)(x-x_0)^{m-1}, \tag{21}$$

and, consequently, $\theta_k^{(n-1)}(y) \neq 0$ on the interval (y_k, y_{k+1}), since $B_0 + B_1 + \cdots + B_k \neq 0$.

6. We can now prove that the function $\Phi(x, y)$ has the same sign as $(-1)^{m-n}$ inside the rectangle D.

First of all, we have

$$\varphi_0^0(x,y) = \frac{(-1)^{m-n}}{(m-1)!(n-1)!} C_0^0 (x-x_0)^{m-1}(y-y_0)^{n-1}, \tag{22}$$

where the coefficient C_0^0 is positive. Hence $\varphi_0^0(x,y)$ has the sign $(-1)^{m-n}$ for $x_0 < x \le x_1,\ y_0 < y \le y_1$.

Next we shall prove that the function $\Phi(x,y)$ has the sign $(-1)^{m-n}$ in the strip $x_0 < x \le x_1,\ y_0 < y < y_n$. To this end consider the function $\theta(y)$, which coincides on each interval $[y_k, y_{k+1}],\ k = 0,1,\dots,n-1$, with the functions $\theta_k(y) = \varphi_0^k(x,y)$, where x has a fixed value in the interval $(x_0, x_1]$. We have proved that $\theta(y)$ is a function continuous on the interval $[y_0, y_n]$, together with its derivatives up to order $n-2$, and that it satisfies the conditions $\theta^j(x_0) = 0,\ \theta^{(j)}(x_n) = 0$ for $j = 0,1,\dots,n-2$. By Rolle's theorem, applied to the function $\theta(y)$ and to the interval $[y_0, y_n]$, the derivative $\theta'(y)$ has at least one zero in the interval (y_0, y_n). This zero is unique.

Indeed, suppose that the derivative $\theta'(y)$ has two zeros in the interval (y_0, y_n). Then, applying Rolle' s theorem successively and taking into account the conditions which the function $\theta(y)$ satisfies at the points y_0 and y_n, we conclude that the derivative $\theta^{(n-2)}(y)$ has $n-1$ zeros in the interval (y_0, y_n). In the intervals $(y_0, y_1]$, $[y_{n-1}, y_n)$ there is not a single zero of $\theta^{(n-2)}(y)$, since we have proved that $\theta_0^{(n-1)}(y)$ and $\theta_n^{(n-1)}(y)$ do not vanish in the intervals (y_0, y_1), (y_{n-1}, y_n). Consequently, the $n-1$ zeros of the derivative $\theta^{(n-2)}(y)$ lie in the interval (y_1, y_{n-1}), but in each interval $(y_k, y_{k+1}]$, where $k = 1,2,\dots,n-2$, there lies only one zero of the function $\theta^{(n-2)}(y)$, since we have proved that the derivative $\theta^{(n-1)}(y)$ does not vanish on the interval (y_k, y_{k+1}). Thus the interval (y_1, y_{n-1}) can contain only $n-2$ zeros of the function $\theta^{(n-2)}(y)$, and we have arrived at a contradiction, which shows that the derivative $\theta'(y)$ has only one zero in the interval (y_0, y_n).

It follows from this that the function $\theta(y)$ has only one extremum in the interval (y_0, y_n), and, since it has sign $(-1)^{m-n}$ in the interval (y_0, y_1), it has sign $(-1)^{m-n}$ in the interval (y_0, y_n). Hence we conclude that the function $\Phi(x,y)$ has sign $(-1)^{m-n}$ in the strip $x_0 < x \le x_1,\ y_0 < y < y_n$.

By the same method, using the properties of the functions $\chi_i(x)$, one can prove that the function $\Phi(x,y)$ has sign $(-1)^{m-n}$ in the strip $x_0 < x < x_m,\ y_k < y \le y_{k+1}$ for each value $k = 0,1,\dots,n-2$, or in the strip $x_0 < x < x_n,\ y_{n-1} < y < y_n$.

The function $\Phi(x,y)$, therefore, has sign $(-1)^{m-n}$ inside the rectangle D.

7. From the preceding property it follows that

$$\left[\begin{array}{c} x_0, x_1, \dots, x_m \\ y_0, y_1, \dots, y_n \end{array} ; f\right] = \frac{\partial^{m+n} f(\xi,\eta)}{\partial x^m \partial y^n} \iint_D \Phi(x,y)\,dx\,dy, \tag{23}$$

where $x_0 < \xi < x_n,\ y_0 < \eta < y_n$. We have

$$\iint_D \Phi(x,y)\,dx\,dy = \frac{(-1)^{m-n}}{m!n!} \tag{24}$$

and, consequently, *the divided difference of order* (m,n) *of the function* $f(x,y)$ *can be represented by the formula*

$$\left[\begin{array}{c} x_0, x_1, \dots, x_m \\ y_0, y_1, \dots, y_n \end{array}; f\right] = \frac{(-1)^{m-n}}{m!n!}\,\frac{\partial^{m+n} f(\xi,\eta)}{\partial x^m \partial y^n}, \tag{25}$$

where $x_0 < \xi < x_n,\ y_0 < y < y_n$.

Hence the estimate follows:

$$\left|\left[\begin{array}{c} x_0, x_1, \dots, x_m \\ y_0, y_1, \dots, y_n \end{array}; f\right]\right| \le \frac{M_{m,n}}{m!n!}, \tag{26}$$

where $M_{m,n}$ is the least upper bound of the absolute value of

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n}$$

in the rectangle $x_0 < x < x_m,\ y_0 < y < y_n$.

Cluj
Romanian People' s Republic

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References

[1] D. V. Ionescu, DAN, **141**, No. 5 (1961). [2] D. V. Ionescu, *Cuadraturi numerice*, București, 1957, chap. III.

Note: Figure translations are in progress. See original paper for figures.