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# Mathematics

K. V. Borozdin

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## Abstract

## Full Text

*Mathematics*

K. V. Borozdin

## A Generalization of Abel' s Theorem

*(Presented by Academician I. M. Vinogradov on XII 1, 1960)*

In the present work two theorems are given which extend a theorem of A. G. Postnikov <sup>(1)</sup>.

**Theorem 1.** Let a convergent series  $\sum_{i=0}^{\infty} a_i = A$  be given, and let  $f(x)$  be such that the function  $xf(x)$  is a continuous function of bounded variation on  $[0, 1]$ . Then

$$\lim_{x \rightarrow 1-0} \sum_{i=0}^{\infty} a_i x^i f(x^i) = Af(1). \quad (1)$$

**Proof.** A continuous function with bounded variation is representable as the difference of two continuous nondecreasing functions. Therefore  $xf(x) = g(x) - h(x)$ , where the functions  $g(x)$  and  $h(x)$  on the interval  $[0, 1]$  are positive, continuous, and nondecreasing. The series  $\sum a_i$  is convergent; consequently, for any  $\varepsilon > 0$  there exists an  $N$  such that, for  $N < n \leq p$ , one always has

$$\left| \sum_{i=n}^p a_i \right| < \varepsilon.$$

The sequences  $\{g(x^i)\}$  and  $\{h(x^i)\}$  do not increase on the interval  $[0, 1]$ . Applying Abel' s lemma to these two sequences (in turn) and to the series  $\sum a_i$  gives

$$|a_n g(x^n) + a_{n+1} g(x^{n+1}) + \dots + a_p g(x^p)| < \varepsilon g(1),$$

$$|a_n h(x^n) + \dots + a_p h(x^p)| < \varepsilon h(1).$$

Consider the sum  $\sum_{i=n}^p a_i x^i f(x^i)$ :

$$|a_n x^n f(x^n) + \dots + a_p x^p f(x^p)| \leq |a_n g(x^n) + \dots + a_p g(x^p)| +$$

$$+|a_n h(x^n) + \dots + a_p h(x^p)| < \varepsilon(g(1) + h(1)).$$

Thus, for the series of continuous functions  $\sum a_i x^i f(x^i)$  on the interval  $[0, 1]$ , the conditions of the Cauchy criterion for uniform convergence are satisfied; consequently,

$$\Phi(x) = \sum_{i=0}^{\infty} a_i x^i f(x^i)$$

is continuous on  $[0, 1]$ . Therefore

$$\lim_{x \rightarrow 1-0} \Phi(x) = \Phi(1) = Af(1),$$

which was required to prove.

**Theorem 2.** Let  $\alpha > 0$  be given, and let

$$\sum_{k=0}^n a_k \sim \frac{n^\alpha}{\Gamma(1+\alpha)}, \quad \text{where } n \rightarrow \infty. \quad (2)$$

Let a function  $f(x)$  be given such that, for some  $\beta \in [0, 1]$ , the expression  $x^\beta f(x)$  represents on the interval  $[0, 1]$  a continuous function of bounded variation. Then

$$\lim_{x \rightarrow 1-0} (1-x)^\alpha \sum_{k=0}^{\infty} a_k x^k f(x^k) = \frac{1}{\Gamma(\alpha)} \int_0^1 f(x) \left(\ln \frac{1}{x}\right)^{\alpha-1} dx. \quad (3)$$

In the proof of this theorem we shall need a lemma.

**Lemma.** Let  $\alpha > 0$  be given, and let  $0 \leq \beta < 1$ ; let the function  $\varphi(x)$  be continuous on  $[0, 1]$ . Then

$$\lim_{x \rightarrow 1-0} (1-x)^\alpha \sum_{k=1}^{\infty} \frac{k^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^k \varphi(x^k) = \frac{1}{\Gamma(\alpha)} \int_0^1 x^{-\beta} \varphi(x) \left(\ln \frac{1}{x}\right)^{\alpha-1} dx. \quad (4)$$

**Proof of the lemma.** For  $|x| < 1$  the expansion

$$\frac{1}{(1-x)^\alpha} = 1 + \alpha x + \frac{\alpha(\alpha+1)}{1 \cdot 2} x^2 + \dots + \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} x^n + \dots \quad (5)$$

is valid. It is easy to establish that, as  $n \rightarrow \infty$ ,

$$\frac{\Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n + 1)} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right). \quad (6)$$

From (6) and (5) it follows that, for the function  $\Phi_\alpha(x) = \sum_{n=1}^{\infty} \frac{n^{\alpha-1}}{\Gamma(\alpha)} x^n$ , where  $x \in [0, 1]$ , the relation

$$\lim_{x \rightarrow 1-0} (1-x)^\alpha \Phi_\alpha(x) = 1 \quad (7)$$

holds.

Consider the one-parameter family of distribution functions  $F_x(u)$ , where the parameter  $x \in (0, 1)$ , defined as follows:

a) at the points  $u = x^n$ , where  $n = 1, 2, \dots$ ,  $F_x(u)$  has jumps equal to

$$\frac{1}{\Phi_\alpha(x^{1-\beta})} \frac{n^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^n;$$

b) at the remaining points of the interval  $u \in (-\infty, \infty)$  the function  $F_x(u)$  is constant and, in particular,  $F_x(0) = 0$ . The jumps of the function  $F_x(u)$  occur only on the interval  $(0, 1)$ ; their sum is

$$\sum_{n=1}^{\infty} \frac{1}{\Phi_\alpha(x^{1-\beta})} \frac{n^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^n = 1.$$

We shall show that  $F_x(u) \rightarrow F(u)$ , where  $F(u)$  is also a distribution function; since outside the interval  $(0, 1)$  all the  $F_x(u)$  coincide, only this interval needs to be investigated.

$$F_x(u) = \sum_{x^n \leq u} \frac{1}{\Phi_\alpha(x^{1-\beta})} \frac{n^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^n = \frac{1}{\Phi_\alpha(x^{1-\beta})\Gamma(\alpha)} \sum_{n \geq \ln u / \ln x} n^{\alpha-1} (x^{1-\beta})^n. \quad (8)$$

It can be proved that, as  $x \rightarrow 1-0$ ,

$$\sum_{n=[\ln u / \ln x + 1]}^{\infty} n^{\alpha-1} (x^{1-\beta})^n = \int_{\ln u / \ln x}^{\infty} t^{\alpha-1} (x^{1-\beta})^t dt + O\left(\left(\ln \frac{1}{x}\right)^{1-\alpha}\right),$$

and, if one takes into account the well-known relations

$$\ln \frac{1}{x} \sim 1-x, \quad 1-x^{1-\beta} \sim (1-\beta)(1-x),$$

then (8) gives

$$F(u) = \lim_{x \rightarrow 1-0} F_x(u) = \frac{(1-\beta)^\alpha}{\Gamma(\alpha)} \int_0^u v^{1-\beta} \left(\ln \frac{1}{v}\right)^{\alpha-1} dv. \quad (9)$$

It is obvious that, for the family  $F_x(u)$ , the conditions of Helly's second theorem (2) are satisfied and, consequently, the following relation holds:

$$\lim_{x \rightarrow 1-0} \int_0^1 \varphi(u) dF_x(u) = \int_0^1 \varphi(u) dF(u). \quad (10)$$

Relation (4) is a consequence of relations (10), (8), and (9).

**Proof of Theorem 2.** The function  $\varphi(x) = x^\beta f(x)$ , as a continuous function of bounded variation, can be represented in the form  $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ , where  $\varphi_1(x)$  and  $\varphi_2(x)$  on the interval  $[0, 1]$  are continuous, nonnegative, and nondecreasing.

It is obvious that, for  $\alpha > 0$  and  $n \rightarrow \infty$ ,

$$\sum_{k=1}^n \frac{k^{\alpha-1}}{\Gamma(\alpha)} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}. \quad (11)$$

Introduce the notation:

$$s_n = \sum_{k=0}^n a_k, \quad \psi_n = \sum_{k=1}^n \frac{k^{\alpha-1}}{\Gamma(\alpha)}. \quad (12)$$

From (2), (11), and (12) it follows that

$$s_n \sim \psi_n. \quad (13)$$

Fix  $\varepsilon > 0$ ; for  $n > N(\varepsilon)$ , by virtue of (13), the following holds:

$$|s_n - \psi_n| < \varepsilon \psi_n. \quad (14)$$

The right-hand sides of (3) and (4) are identical (since  $\varphi(x) = x^\beta f(x)$ ); let us prove the equivalence of the left-hand sides as  $x \rightarrow 1-0$ . Denote

$$W[\varphi_i(x)] = \left| (1-x)^\alpha \sum_{k=0}^{\infty} a_k (x^{1-\beta})^k \varphi_i(x^k) - \right.$$

$$-(1-x)^\alpha \sum_{k=1}^{\infty} \frac{k^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^k \varphi_i(x^k) \Big|;$$

then

$$W[\varphi(x)] \leq W[\varphi_1(x)] + W[\varphi_2(x)], \quad (15)$$

$$\begin{aligned} W[\varphi_1(x)] &= (1-x)^\alpha \left| s_0 \varphi_1(1) + \sum_{k=1}^{\infty} (s_k - s_{k-1}) (x^{1-\beta})^k \varphi_1(x^k) - \right. \\ &\left. - \sum_{k=1}^{\infty} (\psi_k - \psi_{k-1}) (x^{1-\beta})^k \varphi_1(x^k) \right| = (1-x)^\alpha \left| \sum_{k=0}^{\infty} (s_k - \psi_k) \left\{ (x^{1-\beta})^k \varphi_1(x^k) - \right. \right. \\ &\left. \left. - (x^{1-\beta})^{k+1} \varphi_1(x^{k+1}) \right\} \right| \leq (1-x)^\alpha \left| \sum_{k=0}^N (s_k - \psi_k) \left\{ (x^{1-\beta})^k \varphi_1(x^k) - \right. \right. \\ &\left. \left. - (x^{1-\beta})^{k+1} \varphi_1(x^{k+1}) \right\} \right| + \varepsilon (1-x)^\alpha \sum_{N+1}^{\infty} \psi_k \left| (x^{1-\beta})^k \varphi_1(x^k) - (x^{1-\beta})^{k+1} \varphi_1(x^{k+1}) \right|. \end{aligned}$$

On the interval  $[0, 1]$  the function  $\varphi_1(x)$  is nonincreasing. Therefore

$$\begin{aligned} W[\varphi_1(x)] &< (1-x)^\alpha \left| \sum_{k=0}^N (s_k - \psi_k) \left\{ (x^{1-\beta})^k \varphi_1(x^k) - (x^{1-\beta})^{k+1} \varphi_1(x^{k+1}) \right\} \right| + \\ &+ \varepsilon (1-x)^\alpha \psi_{N+1} (x^{1-\beta})^{N+1} \varphi_1(x^{N+1}) + \varepsilon (1-x)^\alpha \sum_{k=1}^{\infty} \frac{k^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^k \varphi_1(x^k). \end{aligned}$$

Here the first two terms on the right-hand side tend to 0 as  $x \rightarrow 1$ , and to the third we apply the lemma (formula (4)):

$$\overline{\lim}_{x \rightarrow 1-0} W[\varphi_1(x)] \leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^1 x^{-\beta} \varphi_1(x) \left( \ln \frac{1}{x} \right)^{\alpha-1} dx = \varepsilon M_1.$$

An analogous inequality is established also for  $W[\varphi_2(x)]$ . Denote  $M = M_1 + M_2$ ; then  $\overline{\lim}_{x \rightarrow 1-0} W[\varphi(x)] \leq \varepsilon M$ , where  $M$  is constant and  $\varepsilon$  is arbitrarily small. Consequently,

$$\begin{aligned} \lim_{x \rightarrow 1-0} (1-x)^\alpha \sum_{k=0}^{\infty} a_k x^k f(x^k) &= \lim_{x \rightarrow 1-0} (1-x)^\alpha \sum_{k=1}^{\infty} \frac{k^{\alpha-1}}{\Gamma(\alpha)} (x^{1-\beta})^k \varphi(x^k) = \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 f(x) \left(\ln \frac{1}{x}\right)^{\alpha-1} dx, \end{aligned}$$

which was required to be proved.

Steklov Mathematical Institute  
Academy of Sciences of the USSR

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## REFERENCES

<sup>1</sup> A. G. Postnikov, *DAN*, **96**, No. 5, 913 (1954). <sup>2</sup> V. I. Glivenko, *The Stieltjes Integral*, Moscow-Leningrad, 1936.

*Note: Figure translations are in progress. See original paper for figures.*

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