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Abstract

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EFFECTIVE INSEPARABILITY OF THE SET OF IDENTICALLY TRUE FORMULAS AND THE SET OF FINITELY REFUTABLE FORMULAS OF CERTAIN ELEMENTARY THEORIES

Let K be some class of models of signature σ. Formulas of the narrow predicate calculus whose nonlogical constants are contained in σ will be called K-formulas. A closed K-formula $\mathfrak{A}$ is called **identically true** on K if it is true on all models of K. $\mathfrak{A}$ is called **finitely refutable** on K if $\mathfrak{A}$ is false on some finite model from K. If $\mathfrak{A}$ is true on all finite K-models, then one says that $\mathfrak{A}$ is **finitely true** on K. By $T(K)$, $F(K)$ we denote, respectively, the totality of all identically true formulas and the totality of finitely refutable K-formulas. In the note [(1)] it was shown that $F(K)$ is not a recursive set if K is the class of all groups or all associative rings, Lie rings, and so on. With the aid of the results of that same note, stronger propositions are proved below, namely that the sets $T(K)$ and $F(K)$ are effectively inseparable if K is one of the above-mentioned classes of groups or rings. Hence, in particular, B. A. Trakhtenbrot’ s theorem [(2)] on the recursive inseparability of the identically true and finitely refutable formulas of the narrow predicate calculus follows directly.

No. 1. Let L denote the class of all algebras over some fixed prime field Γ of prime characteristic π. As in [(2)], we first indicate a regular process by means of which, from each closed L-formula $\mathfrak{A}$, one can construct a new L-formula $\mathfrak{A}^{[n]}$, whose satisfiability will be equivalent to the truth of $\mathfrak{A}$ on all algebras containing fewer than n elements.

Denote by (q, x) the formula $qx = x \;\&\; x^2 = x \;\&\; x \neq 0$, and by $[c, x]$ the formula $cx = x \;\&\; xc = x \;\&\; x^2 = 0$. Elements x for which the formula (q, x) is true will be called q-elements. The totality of those x for which $[c, x]$ is true will be denoted by $\mathfrak{R}_c^*$ and called the subspace belonging to c. The totality of those x for which $px = x$ will be denoted by $\mathfrak{R}_p$. Let us also denote by $\mathfrak{D}(a)$ the conjunction of formulas a1) and a2) from [(1)]. Let now $U(p, q)$ be the conjunction of the formulas

$$(xy)((q, x) \;\&\; [x, y] \rightarrow py = y) \;\&\; (xy)(px = x \;\&\; py = y \rightarrow xy = 0);$$

$$(xy)[(q,x) \;\&\; (q,y) \;\&\; x \neq y \to (\exists u)([x,u] \vee [y,u] \;\&\; \neg[x,u] \vee \neg[y,u])];$$

$$(x)(\exists y)[px = x \to (q,y) \;\&\; (u)([y,u] \leftrightarrow u = x \vee u = 2x \vee \cdots \vee u = \pi x)];$$

$$(xy)(\exists z)[(q,x) \;\&\; (q,y) \to (q,z) \;\&\; (u)([z,u] \leftrightarrow$$

$$\leftrightarrow (\exists vw)([x,v] \;\&\; [y,w] \;\&\; u = v + w))].$$

The truth of $U(p,q)$ in some algebra $\mathfrak{R} \in L$ containing the elements p, q means that: 1) the subspace belonging to any q-element is contained in $\mathfrak{R}_p$; 2) the subspaces belonging to distinct q-elements are distinct; 3) every zero-dimensional or one-dimensional subspace,

of the space $\mathfrak{R}_p$ belongs to some q-element; 4) the product of any elements of $\mathfrak{R}_p$ is equal to zero. From $U(p,q)$ it follows that the space belonging to a q-element is a linear subspace in $\mathfrak{R}_p$, and that every finite-dimensional subspace of $\mathfrak{R}_p$ belongs to some q-element.

No. 2. It is easy to compute that the number ρ of distinct algebras that can be constructed by introducing, in various ways, the operation of multiplication into a given linear n-dimensional space over Γ, is equal to π^{n^3}. Let $t = F(n)$ be the number of distinct linear subspaces of such a space, and let $n = G(t)$ be the inverse function, which for definiteness we put equal to 0 if t does not belong to the set of values of F. From the explicit formula for $F(n)$ it is clear that $F(n)$ and $G(t)$ are primitive recursive functions. Therefore the function $S(t) = \pi^{G(t)^3}$ will also be primitive recursive. By the method set out in (1), we construct a formula $\mathfrak{S}(a,b)$ having the following properties: 1) if in some algebra $\mathfrak{R}$ elements a, b satisfying $\mathfrak{S}(a,b)$ are distinguished, and if the number of a-elements in $\mathfrak{R}$ is equal to t, then the number of b-elements in $\mathfrak{R}$ is equal to $S(t)$; 2) for every t there exists a finite algebra $\mathfrak{R}$ with identity, having elements a, b with the property $\mathfrak{S}(a,b)$, in which the number of a-elements is equal to t.

No. 3. Denote by $V(c,q,g,a,b)$ the formula

$$\mathfrak{S}_g(a,b) \;\&\; (xy)\,(gx = x \;\&\; gy = y \to g \cdot xy = xy) \;\&$$

$$\&\; (\exists z)\{(y)\,[(q,y) \;\&\; \mathfrak{R}_y^* \subseteq \mathfrak{R}_c^* \to g \cdot zy = zy \;\&\; (a, zy)] \;\&$$

$$\&\; (x)\,[(a,x) \;\&\; gx = x \to (\exists y)\,((q,y) \;\&\; \mathfrak{R}_y^* \subseteq \mathfrak{R}_c^* \;\&\; x = zy)] \;\&$$

$$\& \ (xy)\,[(q,x) \ \& \ \mathfrak{R}^*_x \subseteq \mathfrak{R}^*_c \ \& \ (q,y) \ \& \ \mathfrak{R}^*_y \subseteq \mathfrak{R}^*_c \ \& \ zx = zy \to x = y]\},$$

where $\mathfrak{S}_g(a,b)$ denotes the relativization of the formula $\mathfrak{S}(a,b)$ to the set $\mathfrak{R}_g$, and $\mathfrak{R}^*_y \subseteq \mathfrak{R}^*_c$ denotes the formula $(u)([y,u] \to [c,u])$.

Suppose that in the algebra $\mathfrak{R}$ elements p,c,q,g,a,b are distinguished for which $U(p,q) \ \& \ V(c,q,g,a,b)$ is true. Then the space $\mathfrak{R}_g$ will be a subalgebra containing the elements a,b, which inside $\mathfrak{R}_g$ have the property $\mathfrak{S}(a,b)$. In this case the number of a-elements in $\mathfrak{R}_g$ will be equal to the number of q-elements y for which $\mathfrak{R}^*_y$ is contained in $\mathfrak{R}^*_c$, and hence the number of b-elements in $\mathfrak{R}_g$ is equal to π^{r^3}, where r is the dimension of $\mathfrak{R}^*_c$.

Denote also by $W(c,g,b)$ the conjunction of the formulas

$$(xyz)\,[gx = x \ \& \ (b,x) \ \& \ [c,y] \ \& \ [c,z] \to (xy = xz \to y = z) \ \&$$

$$\& \ (\exists u)\,([c,u] \ \& \ xy \cdot xz = xu)];$$

$$(xy)\,[gx = x \ \& \ (b,x) \ \& \ gy = y \ \& \ (b,y) \ \& \ x \neq y \to$$

$$\to (\exists uvww_1)\,([c,u] \ \& \ [c,v] \ \& \ [c,w] \ \& \ [c,w_1] \ \& \ xu \cdot xv = xw \ \&$$

$$\& \ yu \cdot yv = yw_1 \ \& \ w \neq w_1)].$$

The truth of the formula $W(c,g,b)$ in an algebra $\mathfrak{R}$ containing distinguished elements c,g,b means that, for any fixed b-element x of the subalgebra $\mathfrak{R}_g$, the elements of the form xu $(u \in \mathfrak{R}^*_c)$ form a subalgebra $\mathfrak{R}^x_c$, isomorphic to a special algebra that is obtained if, in the linear space $\mathfrak{R}^*_c$, one introduces the multiplication operation $\otimes$ by means of the condition $y \otimes z = u \leftrightarrow xy \cdot xz = xu$.

Suppose that in $\mathfrak{R}$ the axioms $U, V, W(p,g,b)$ are satisfied, and let the dimension of $\mathfrak{R}_p$ be n. Then the number of b-elements x will be equal to π^{n^3}, i.e. it will be equal to the number of all possible multiplications on $\mathfrak{R}_p$. Therefore, among the subalgebras $\mathfrak{R}^x_p$ there will be subalgebras isomorphic to any prescribed algebra of dimension n.

No. 4. Let $\mathfrak{A}$ be any closed formula of the narrow predicate calculus relating to rings. Denote by $\mathfrak{A}(c,x)$ the relativiza-

$\mathcal{A}$ relative to $\mathfrak{R}^x_c$, i.e. relative to the set of elements u with the property

$$P(u,c,x) \ \overset{df}{=} \ (\exists y)((c,y) \ \& \ u = xy).$$

Put also

$$E(c,x) \overset{df}{=} (\exists e)(P(e,c,x) \mathbin{\&} (u)(P(u,c,x) \to ue = u \mathbin{\&} eu = u)),$$

$$Q(p,n) \overset{df}{=} (\exists x_1 \ldots x_n)(px_1 = x_1 \mathbin{\&} \ldots \mathbin{\&} px_n = x_n \mathbin{\&} \bigwedge a_1 x_1 + \ldots + a_{nx} n \neq 0),$$

where the conjunction $\bigwedge$ ranges over all possible combinations of the numbers $a_1, \ldots, a_n$, distinct from the combination $0, \ldots, 0$ ($a = 0, 1, \ldots, \pi - 1$). The truth of $Q(p,n)$ means that $\mathfrak{R}_p$ has dimension not less than n.

Finally, introduce the notations

$$R(p,q) \overset{df}{=} U(p,q) \mathbin{\&} \mathfrak{D}(q) \mathbin{\&} (c)((q,c) \to (\exists gab)\, X(c,q,g,a,b)),$$

$$\mathfrak{A}(q) \overset{df}{=} (cgabx)((q,c) \mathbin{\&} X(c,q,g,a,b) \mathbin{\&} gx = x \mathbin{\&} (b,x) \mathbin{\&} E(c,x) \to \mathfrak{A}(c,x)),$$

where $X(c,q,g,a,b)$ is the conjunction of the formulas $V(c,q,g,a,b)$, $W(c,g,b)$.

Lemma 1. If the formula $\mathfrak{A}$ is true on all algebras with identity whose dimension is less than n, then the formula

$$\neg Q(p,n) \mathbin{\&} R(p,q) \to \mathfrak{A}(q)$$

is true on all algebras.

Indeed, let in some algebra $\mathfrak{R}$ elements p, q, c, g, a, b, x be distinguished with the properties $\neg Q(p,n)$, $R(p,q)$, (q,c), $X(c,q,g,a,b)$, $gx = x$, (b,x), $E(c,x)$. Then the algebra $\mathfrak{R}_c^x$ will have dimension equal to the dimension of $\mathfrak{R}_c$, i.e. will have dimension less than n. Moreover, $\mathfrak{R}_c^x$ will have an identity and, consequently, $\mathfrak{A}(c,x)$ will be a true formula.

Lemma 2. If $\mathfrak{A}$ is false on some algebra of finite dimension n having an identity, then the formula

$$Q(p,n) \mathbin{\&} R(p,q) \to \neg\mathfrak{A}(q)$$

is true on all algebras.

Let in an arbitrary algebra $\mathfrak{R}$ elements p, q be distinguished so that $Q(p,n)$ and $R(p,q)$ are true. Then in $\mathfrak{R}_p$ there is an n-dimensional subspace which belongs to some q-element c. Therefore in $\mathfrak{R}$ there exist g, a, b for which $\mathfrak{R}_g$ will be a subalgebra containing a, b and having π^{n^3} b-elements. Shifting $\mathfrak{R}_c^*$ by means of these b-elements, we obtain π^{n^3} subalgebras $\mathfrak{R}_c^x$, among which there will also be a subalgebra with identity in which the formula $\mathfrak{A}$ is false. Thus, in $\mathfrak{R}$ the formula $\neg\mathfrak{A}(q)$ is true.

Lemma 3. If the closed formula $\mathfrak{A}$ pertaining to rings with identity is finitely refutable, and the formula $\mathfrak{B}$ is finitely true, then the formula

$$(pq)\,[R(p,q) \to (\mathfrak{A}(q) \to \mathfrak{B}(q))] \tag{1}$$

is true on all algebras.

Let $\mathfrak{A}$ be false in some n-dimensional algebra with identity. Then, according to Lemmas 1 and 2, the formulas

$$\neg Q(p,n)\ \&\ R(p,q) \to \mathfrak{B}(q),$$

$$Q(p,n)\ \&\ R(p,q) \to \mathfrak{A}(q)$$

are true on all algebras, and, consequently, formula (1) is true on all algebras.

Lemma 4. If the formula $\mathfrak{B}$ is false in a suitable finite algebra with identity, and the formula $\mathfrak{A}$ is true in all finite algebras with identity, then formula (1) is false in a suitable finite algebra with identity.

To prove this lemma, one must, knowing an n-ary algebra $\mathfrak{A}$ with identity on which $\mathfrak{B}$ is false, construct a finite-dimensional algebra with elements p, q, on which $R(p,q)$ and $\mathfrak{A}(q)$ would be true, while $\mathfrak{B}(q)$ would be false. The construction of such an algebra is analogous to the construction indicated in (1). In view of its somewhat cumbersome nature, we omit it here.

No. 5. Let A^1, A^2 be arbitrary disjoint recursively enumerable sets of natural numbers. In (1) an effective method was indicated which, from Post-Kleene numbers of the sets A^1, A^2, makes it possible to construct sequences of formulas $\mathfrak{A}_j^\lambda$ $(\lambda = 1, 2; j = 0, 1, 2, ...)$, concerning rings and such that $m \in A^\lambda$ if and only if $\mathfrak{A}_m^\lambda$ is false on some finite algebra with identity. Consider the sequence of formulas

$$\Phi_m \stackrel{df}{=} (pq)\,[R(p,q) \to (\mathfrak{A}_m^1(q) \to \mathfrak{A}_m^2(q))].$$

According to Lemmas 3, 4, if $m \in A^1$, then Φ_m is identically true, while if $m \in A^2$, then Φ_m is finitely refutable in the class of algebras with identity. In other words, every pair of disjoint recursively enumerable sets A^1, A^2 is recursively reducible to the pair of sets $T(L), F(L)$. Taking as A^1, A^2 an effectively inseparable pair, or using Muchnik' s theorem ([3]), we directly conclude that the pair $T(L), F(L)$ is also effectively inseparable. Thus we have proved:

Theorem 1. *The set of identically true formulas and the set of finitely refutable formulas on the class of all rings of prime characteristic with identity are effectively inseparable.*

Using, as in ([1]), the correspondence between rings and groups, we arrive at the conclusion that, together with Theorem 1, the following is also true:

Theorem 2. *The sets of formulas identically true and finitely refutable on the class of all metabelian groups with the identity $x^\pi = 1$, and also on the class of all rings of characteristic $\pi = 3, 5, \ldots$ with the identities $xy = -yx,\ x{\cdot}yz = xy{\cdot}z = 0$, are effectively inseparable.*

It follows from Theorem 2 that the indicated sets of formulas are effectively inseparable also for the class of all groups, the class of all associative rings, the class of all Lie rings, etc.

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Note: Figure translations are in progress. See original paper for figures.