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Abstract

Full Text

MATHEMATICS

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SOME QUESTIONS ON THE COMPLETENESS OF SYSTEMS

(Presented by Academician S. N. Bernstein, XI 3, 1960)

In the present paper we consider questions connected with completeness in linear spaces, when not only the magnitudes of the deviations of the approximating aggregates are taken into account, but also the magnitudes of the coefficients of these aggregates. Davis and Fan-Ci noted for the first time the possibility of a general treatment of these questions ⁽¹⁾ (see also ⁽²⁾); their results are special cases of some of the theorems given below. We arrived at the general theory set forth in the present paper starting from our more concrete results connected with extremal problems with additional conditions for polynomials and moments ⁽³⁾, extremal problems for analytic functions satisfying additional restrictions ⁽⁴⁾, and, finally, questions of approximation on sets of analytic capacity zero ⁽⁵⁾. The main instrument for proving the theorems of this paper is Theorem 1 from paper ⁽³⁾ (a generalization of the latter theorem is Theorem 2 of paper ⁽⁴⁾).

Let E be a linear locally convex topological space (real or complex); let $p(f)$ be a continuous symmetric convex functional on E (a seminorm). Further, let $p_1(\lambda_1, \dots, \lambda_n, \dots)$ be a function of a countable set of numerical arguments $\lambda_1, \dots, \lambda_n, \dots$ (real or complex, according to what E is). Suppose that for every n the function

$$p_1(\lambda_1, \dots, \lambda_n) = p_1(\lambda_1, \dots, \lambda_n, 0, \dots, 0, \dots)$$

is a continuous, convex, symmetric functional in the Euclidean space $E_n = \{(\lambda_1, \dots, \lambda_n)\}$. Let $\{\varphi_n\}$ be a sequence of elements of E and let $\omega \in E$.

Theorem 1. In order that, for every $\varepsilon > 0$, it be possible to choose coefficients $\lambda_1^*, \dots, \lambda_N^*$ (N also depends on ε) such that the inequalities

$$p\left(\omega - \sum_1^N \lambda_j^* \varphi_j\right) < \varepsilon, \quad p_1(\lambda_1^*, \dots, \lambda_N^*) < \varepsilon, \quad (1)$$

hold, it is necessary and sufficient that for every linear functional $l(f) \in E^*$ (E^* is the space conjugate to E) satisfying the inequalities

$$|l(f)| \leq p(f), \quad f \in E; \quad \left| \sum_1^n \lambda_j l(\varphi_j) \right| \leq p_1(\lambda_1, \dots, \lambda_n) \quad (2)$$

for all n ; $\lambda_1, \dots, \lambda_n$, one have

$$l(\omega) = 0. \quad (3)$$

Proof. Let us prove necessity, i.e., that (3) follows from (1) and (2). We have:

$$\begin{aligned} |l(\omega)| &\leq \left| l\left(\omega - \sum_{j=1}^N \lambda_j^* \varphi_j\right) \right| + \left| l\left(\sum_{j=1}^N \lambda_j^* \varphi_j\right) \right| \leq \\ &\leq p\left(\omega - \sum_{j=1}^N \lambda_j^* \varphi_j\right) + p_1(\lambda_1^*, \dots, \lambda_N^*) < 2\varepsilon. \end{aligned}$$

Let us prove sufficiency, i.e., that if (3) follows from (2), then (1) also holds. Let B^n be the set of those functionals $L \in E^*$ for which (2) holds for the given n . Put

$$\delta_n = \sup_{L \in B^n} |L(\omega)| \quad (4)$$

and prove that $\delta_n \rightarrow 0$. Let $L_n \in B^n$ be an extremal functional in (4): $|L_n(\omega)| = \delta_n$. Introduce the space $E_1 \in E$, the linear span of the system $\{\omega; \varphi_n\}$, and let $\{\psi_m\}$ be a countable set everywhere dense (in the seminorm p) in E_1 . For any m the set $\{L_n(\psi_m)\}$ is bounded, and therefore from the sequence $\{L_n\}$ one can choose, by the diagonal process, a subsequence converging to a definite limit $L(\psi_m)$ for each m . We shall assume that $\{L_n\}$ already has this property. Then for any $\psi \in E_1$ we have: $L(\psi) = \lim_{n \rightarrow \infty} L_n(\psi)$ exists. The functional $L(\psi)$, defined on E_1 , is additive and homogeneous. Moreover,

$$|L(\psi)| = \lim_{n \rightarrow \infty} |L_n(\psi)| \leq p(\psi), \quad \psi \in E_1; \quad (5)$$

$$\left| \sum_1^m \lambda_{jL}(\varphi_j) \right| = \lim_{n \rightarrow \infty} \left| \sum_1^m \lambda_{jL} n(\varphi_j) \right| \leq p_1(\lambda_1, \dots, \lambda_m). \quad (6)$$

Extending L from E_1 to all of E , we arrive at a functional satisfying (2), but for which $|L(\omega)| = \lim |L_n(\omega)| = \lim \delta_n \neq 0$. This contradicts the assumption. Thus, $\delta_n \rightarrow 0$. But, according to Theorem 1, from (3) we have

$$\delta_n = \inf_{(\lambda_1, \dots, \lambda_n)} \left[p \left(\omega - \sum_1^n \lambda_j \varphi_j \right) + p_1(\lambda_1, \dots, \lambda_n) \right].$$

The theorem is proved.

Theorem 1 makes the following definitions natural:

Definition 1. The system $\{\varphi_n\}$ is **complete** (p, p_1) in E if, for every $l \in E^*$, (3) follows from (2) for every $\omega \in E$. The system $\{\varphi_n\}$ is **closed** (p, p_1) in E if, for any $\omega \in E$ and arbitrary $\varepsilon > 0$, one can find coefficients $\lambda_1^*, \dots, \lambda_N^*$ in finite number for which (1) is satisfied.

From Theorem 1 it follows immediately:

Theorem 2. In order that the system $\{\varphi_n\}$ be complete (p, p_1) in E , it is necessary and sufficient that it be closed (p, p_1) in E .

Remark. When $p_1(\lambda_1, \dots, \lambda_n) \equiv 0$, Theorem 1, if E is a Banach space, gives the classical criterion for an element ω to belong to the closure of the linear span of the system $\{\varphi_n\}$. Under the same conditions, Theorem 2 expresses the usual relation between ordinary completeness and closedness.

If, further, E is a Banach space, and

$$p_1 = \sum_1^\infty \varepsilon_j |\lambda_j| \quad (\varepsilon_j > 0 \text{ are given})$$

or

$$p_1 = \left[\sum_1^\infty |\lambda_j|^p \right]^{1/p}, \quad p > 1,$$

then Theorem 2 gives the principal result of the works ^(1,2).

Definition 2. The system $\{\varphi_n\}$ is called **completely complete** (p, p_1) in E if, for every m , the "truncated" system $\{\varphi_n\}_{n=m}^\infty$ is complete $(p, p_1^{(m)})$, where

$$p_1^{(m)}(\lambda_1, \dots, \lambda_n, \dots) = p_1(0, \dots, 0, \lambda_1, \dots, \lambda_n, \dots).$$

Theorem 3. If $\{\varphi_n\}$ is a completely complete (p, p_1) system in E , then for an arbitrarily given $\varepsilon_0 > 0$ there exists a universal series

$$\sum_1^\infty \lambda_j \varphi_j, \tag{7}$$

having the following properties:

1.

$$p_1(\lambda_1, \dots, \lambda_n) < \varepsilon_0, \quad n = 1, 2, \dots \quad (8)$$

2. For every $\omega \in E$ there are two sequences of indices $\{n_k\}$ and $\{m_k\}$, $n_k < m_k$, $n_k \rightarrow \infty$, such that

$$\lim_{k \rightarrow \infty} p \left(\omega - \sum_{j=n_k+1}^{m_k} \lambda_j \varphi_j \right) = 0, \quad \lim_{k \rightarrow \infty} p_1(0, \dots, 0, \lambda_{n_k+1}, \dots, \lambda_{m_k}) = 0. \quad (9)$$

Theorem 4. Under the conditions of Theorem 3 there exists a series (7), having property (8) and such that for every $\omega \in E$ there is a subsequence of partial sums of this series converging to ω in the seminorm p .

The following theorem has a somewhat different character:

Theorem 5. Suppose that from the inequalities

$$\left| \sum_{j=1}^n \lambda_j l(\varphi_j) \right| \leq p_1(\lambda_1, \dots, \lambda_n), \quad (10)$$

holding for all n and $\lambda_1, \dots, \lambda_n$, it follows for every $l \in E^*$ that

$$|l(f)| \leq p(f), \quad f \in E. \quad (11)$$

Then for every $\omega \in E$ we have, for all $K \geq 1$,

$$\sigma = \sup |l(\omega)| = \inf_{\lambda_1, \dots, \lambda_n} \left[K p \left(\omega - \sum_1^n \lambda_j \varphi_j \right) + p_1(\lambda_1, \dots, \lambda_n) \right], \quad (12)$$

where the supremum is taken over all $l \in E^*$ satisfying (10).

Corollary.

$$\sigma = \inf \lim_{k \rightarrow \infty} p_1(\lambda_1^k, \dots, \lambda_{n_k}^k),$$

where the infimum is taken over all possible systems $(\lambda_1^k, \dots, \lambda_{n_k}^k)$ such that

$$\lim_{k \rightarrow \infty} p \left(\omega - \sum_1^{n_k} \lambda_j^k \varphi_j \right) = 0.$$

Let us dwell on some applications of the theorems stated above.

Theorem 6. Suppose that the sequences $\{\alpha_\nu\}$ and $\{\beta_\nu\}$ satisfy the conditions:

$$|\alpha_\nu| < 1, \quad \nu = 1, 2, \dots; \quad \sum_1^\infty (1 - |\alpha_\nu|) = \infty; \quad \frac{|\alpha_\nu| - |\alpha_{\nu-1}|}{(1 - |\alpha_\nu|)(1 - |\alpha_{\nu-1}|)} \geq d > 0,$$

$$|\beta_\nu| > 1, \quad \nu = 1, 2, \dots; \quad \sum_1^\infty (|\beta_\nu| - 1) = \infty; \quad \frac{|\beta_{\nu-1}| - |\beta_\nu|}{(1 - |\beta_\nu|)(1 - |\beta_{\nu-1}|)} \geq d > 0,$$
(13)

where d does not depend on ν . Let $\{r_\nu\}$ and $\{R_\nu\}$ be two sequences of positive numbers for which

$$\lim_{\nu \rightarrow \infty} r_\nu = \lim_{\nu \rightarrow \infty} R_\nu = \infty. \quad (14)$$

For any function $\omega(\theta)$ belonging to $L_p[0, 2\pi]$, $p \geq 1$, or to $C[0, 2\pi]$, and any $\varepsilon > 0$, one can find numbers $\lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n$ such that

$$\left\| \omega(\theta) - \sum_1^n \frac{\lambda_\nu}{e^{i\theta} - \alpha_\nu} - \sum_1^n \frac{\mu_\nu}{e^{i\theta} - \beta_\nu} \right\| < \varepsilon,$$

$$\sum_1^n e^{-\frac{r_\nu}{1 - |\alpha_\nu|}} |\lambda_\nu| + \sum_1^n e^{-\frac{R_\nu}{1 - |\beta_\nu|}} |\mu_\nu| < \varepsilon.$$

If all $\alpha_\nu > 0$ and $\beta_\nu > 0$, then for every $t > 0$, in any of our spaces there exist functions $\omega(\theta)$ for which:

$$\inf_{\{\lambda_\nu\}, \{\mu_\nu\}} \left[\left\| \omega(\theta) - \sum_1^n \frac{\lambda_\nu}{e^{i\theta} - \alpha_\nu} - \sum_1^n \frac{\mu_\nu}{e^{i\theta} - \beta_\nu} \right\| + \sum_1^n e^{-\frac{t}{1 - |\alpha_\nu|}} |\lambda_\nu| + \sum_1^n e^{-\frac{t}{1 - |\beta_\nu|}} |\mu_\nu| \right] \geq \varepsilon_0,$$

where ε_0 , for the given t , depends only on ω .

The proof of Theorem 6 is based on Theorem 2 and the uniqueness theorem from (4)*.

Developing Theorem 6 further, one can, with the aid of Theorem 4, obtain the following result:

Theorem 7. There exists a series

$$\sum_1^\infty \frac{\lambda_\nu}{e^{i\theta} - \alpha_\nu} + \sum_1^\infty \frac{\mu_\nu}{e^{i\theta} - \beta_\nu}, \quad (15)$$

satisfying the condition

$$\sum_1^{\infty} e^{-\frac{r_\nu}{1-|\alpha_\nu|}} |\lambda_\nu| + \sum_1^{\infty} e^{-\frac{R_\nu}{1-|\beta_\nu|}} |\mu_\nu| < \varepsilon_0 \quad (16)$$

($\{\alpha_\nu\}$, $\{\beta_\nu\}$, $\{r_\nu\}$, $\{R_\nu\}$ are as in Theorem 6; ε_0 is given arbitrarily) and such that, for every $\omega(\theta) \in L_p[0, 2\pi]$, $p \geq 1$, or $\omega(\theta) \in C[0, 2\pi]$, there exists a subsequence of partial sums of the series (15) converging (in the norm of the space) to ω . If $\alpha_\nu > 0$, $\beta_\nu > 0$, then for any series (15) having the last property it must be that

$$\sum_1^{\infty} e^{-\frac{t}{1-|\alpha_\nu|}} |\lambda_\nu| = \infty; \quad \sum_1^{\infty} e^{-\frac{t}{1-|\beta_\nu|}} |\mu_\nu| = \infty$$

for every $t > 0$.

As an application of Theorem 5, the entire main content of paper (5) can be obtained.

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CITED LITERATURE

¹ F. Davis, Ky Fan, *Due Math. J.*, **24**, No. 2, 182 (1957). ² Ky Fan, *Seminars on Analytic Functions Institute for Advanced Study, Princeton*, **2**, 1958, p. 202. ³ S. Ya. Khavinson, *DAN*, **130**, No. 5, 997 (1960). ⁴ S. Ya. Khavinson, *DAN*, **135**, No. 2 (1960). ⁵ S. Ya. Khavinson, *DAN* **131**, No. 1, 44 (1960). ⁶ S. Ya. Khavinson, *Abstracts of reports, V Conference on the Theory of Functions of a Complex Variable, Yerevan, 1960*. ⁷ I. V. Ushakova, *DAN*, **130**, No. 1, 29 (1960).

* I take this opportunity to note that the statement made in (4) and (6) that the uniqueness theorem cited in (4) generalizes the theorem of I. V. Ushakova (7) is incorrect. The uniqueness theorem from (4) and the uniqueness theorem of I. V. Ushakova (7) do not completely include one another.

Note: Figure translations are in progress. See original paper for figures.

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