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Abstract

Full Text

MATHEMATICS

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ON THE COMPLETENESS OF THE ZEROS OF ENTIRE FUNCTIONS

(Presented by Academician L. S. Pontryagin on 3 VII 1961)

1°. In this paper it is shown that for the set of zeros of one important class of entire functions of exponential type one can introduce a characteristic playing the same role as the degree of a polynomial does for a polynomial. The terminology and notation are the same as in the note ⁽¹⁾.

2°. **Definition.** A monotone sequence of real points $\{\xi_\nu\}$, $\nu = \pm 1, \pm 2, \dots$, will be called regular with characteristic $(\sigma; p)$ if the points ξ_ν admit the representation

$$\xi_\nu = (\nu + \delta(\nu)) \frac{\pi}{\sigma} \quad \text{for } \nu < 0; \quad \xi_\nu = (\nu - \delta(\nu)) \frac{\pi}{\sigma} \quad \text{for } \nu > 0, \quad (1)$$

where the shifts $\delta(\nu)$ satisfy the conditions

$$-\infty < -a \leq \delta(\nu) \leq b < +\infty, \quad |\nu(\delta(\nu+1) - \delta(\nu))| < c < +\infty, \quad (2)$$

and p is the smallest integer for which the quantity

$$\chi(N) = \sum_{\nu=1}^N \frac{\delta(-\nu) + \delta(\nu) - 1}{\nu} + 2p \ln N \quad (3)$$

is bounded below. If $\chi(N)$ is also bounded above, then the sequence $\{\xi_\nu\}$ will be called completely regular.

Lemma (see ⁽¹⁾). Let $\{\xi_\nu\}$ be a regular sequence with characteristic $(\sigma; p)$, and let $v(z; \{\xi_\nu\})$ be the function with zeros at the points ξ

$$v(z; \{\xi_\nu\}) = \prod_{\nu=1}^{\infty} \left(1 + \frac{\sigma z - \delta(-\nu)\pi}{\nu\pi} \right) e^{-\frac{\sigma z - \delta(-\nu)\pi}{\nu\pi}} \times \\ \times \prod_{\nu=1}^{\infty} \left(1 - \frac{\sigma z + \delta(\nu)\pi}{\nu\pi} \right) e^{\frac{\sigma z + \delta(\nu)\pi}{\nu\pi}}. \quad (4)$$

The function $v(z; \{\xi\})$ has the following properties:

- 1) $v(z)$ is a function of type σ with indicator $h(\varphi) = \sigma |\sin \varphi|$.
- 2) On the real axis

$$v(x; \{\xi\}) = (\xi_j^- - x) \cdots (\xi_1^- - x)(\xi_1^+ - x) \cdots (\xi_j^+ - x) B(x; \{\xi\}) \times \exp \left(\sum_{\nu=1}^{[\sigma|x|/\pi]} \frac{\delta(-\nu) + \delta(\nu) - 1}{\nu} \right), \quad (5)$$

where $j = \max([a] + 1, [b] + 1)$; ξ^- , ξ^+ are the points ξ_ν nearest to x on the left and on the right; $B(x)$ is positive and bounded on both sides

$$0 < \alpha(a, b, c) \leq B(x; \{\xi\}) \leq \beta(a, b, c) < +\infty. \quad (6)$$

- 3) The ratio $v'(x') : v(x'')$, where the points x' , x'' lie outside the ε -neighborhoods of the points ξ_ν , is bounded in modulus from above by a constant $\gamma = \gamma(\varepsilon, a, b, c)$;

$$|v(x'')| > d(|x''| + 1)^{-2p}; \quad d = d(\varepsilon, a, d, c) > 0.$$

The sequence $\{\eta_\nu\}$,

$$\eta_\nu = \frac{\xi_{\nu+1} + \xi_\nu}{2}$$

is also regular, with characteristic (σ, p) . * Introduce the functions

$$u(z; \{\xi\}) = v(z; \{\eta\}), \quad v_1(z) = R_{2p}(z)v(z), \quad u_1(z) = Q_{2p}(z)u(z),$$

where $R_{2p}(z)$ and $Q_{2p}(z)$ are arbitrary real polynomials of degree $2p$ with complex zeros, assuming values of different sign on the real axis. According to Theorem VI.18 from (2), the function $\omega(z) = u_1(z) + iv_1(z)$ belongs to the class B (see, for example, (2, 3)) with p zeros in the upper half-plane.

Theorem 1. Let $f(z)$ be a real entire function of exponential type not exceeding σ , $L_f < \infty$, and let all points of a $(\sigma; p)$ -regular sequence $\{\xi\}$ be zeros of the function $f(z)$. If the function $f(z)$ does not vanish identically, then $f(z)$ has no more than $2p$ zeros distinct from the points $\{\xi\}$.

Proof. Let the number of zeros of the function $f(z)$ distinct from $\{\xi\}$ be greater than $2p$. Choose $2p$ such zeros and construct a function $\hat{v}_1(z; \{\xi\})$ in which $2p$ arbitrary complex zeros are replaced by these zeros. Obviously,

$$\varphi(z) = f(z) : \hat{v}_1(z)$$

is an entire function of no more than minimal type. If $\varphi(z)$ does not reduce to a constant, then $\lim |\varphi(x)| = \infty$ as $x \rightarrow \infty$ on a set of positive relative density and, by virtue of (4) and (5), L_f cannot be bounded.

This theorem is an analogue of the property that the number of roots of a polynomial coincides with its degree. Especially important is the special case of the theorem for $p = 0$.

3°. Let $\{a\}$ be the sequence of zeros of some function $f(z)$. If from this sequence one can remove q points so that the remaining zeros will alternate with the sequence $\{\xi\}$, then we shall say that the number of nontrivial zeros of the function $f(z)$ relative to the sequence $\{\xi\}$ is equal to d .

Theorem 2. *Let $f(z)$ be a real entire function of exponential type $\leq \sigma$ and $L_f < +\infty$.** If at all points of some $(\sigma; p)$ -regular sequence $\{\xi\}$, for which the function*

$$u(z) = v_1 \left(z; \left\{ \frac{\xi_\nu + \xi_{\nu+1}}{2} \right\} \right)$$

the inequality

$$f(\xi) : u_1(\xi) \leq 1, \tag{7}$$

is satisfied, then all zeros of the difference $u(z) - f(z)$, with the possible exception of $2p$ zeros, are real and alternate with the points ξ .

Proof. Let T be the closure of the image of the real axis under the mapping $t = \omega(z) : f(z)$ ($\omega(z) = u_1(z) + iv_1(z)$). We shall prove that the half-closed interval $0 \leq t < 1$ lies inside one of the domains D complementary to T . Suppose the contrary: let $t_0 \in T$, $0 \leq t_0 < 1$. Then there exists a sequence of real points $\{x_n\}$ for which

$$\lim(\omega(x_n) : f(x_n)) = t_0$$

or

$$\frac{f(x_n)}{u(x_n)} \rightarrow t_0^{-1} > 1, \quad \frac{v(x_n)}{f(x_n)} \rightarrow 0.$$

Consequently, $x_n \rightarrow \infty$, since otherwise $x_n \rightarrow \xi$, which would contradict inequality (7). From $v(x_n) \rightarrow 0$ and relation (4) it follows that, for sufficiently large n , x_n is arbitrarily close to $\xi_-(x)$ or $\xi_+(x)$ —denote this point by $\xi(x)$.

* For $\nu = 1$, $\eta_1 = \frac{\xi_{-1} + \xi_1}{2}$.

** It is sufficient to require boundedness in modulus of the ratio $f'(x) : u(\xi)$, when x belongs to some system of ε' -neighborhoods of ξ_ν , and boundedness of

$$f(x) \exp \left(- \sum_{\nu=1}^{\lfloor |x|/\sigma/\pi \rfloor} \frac{\delta(-\nu) + \delta(\nu) - 1}{\nu} \right).$$

For $L_f < +\infty$ this condition is certainly satisfied.

From the lemma it follows that $\frac{f(x_n)}{u(x_n)} - \frac{f(\xi)}{u(\xi)} \rightarrow 0$, which is impossible. From the fact that the interval $0 \leq t < 1$ belongs to D , by Theorem 2 of ⁽¹⁾ it follows that the function $\omega_t(z) = [u_1(z) - tf(z)] + iv_1(z)$ belongs to the class B and has p zeros in the upper half-plane, while the component $u_1(z) - tf(z)$ has no more than $2p$ nontrivial zeros. This property, by continuity, is preserved as $t \rightarrow 1$.

Remark. If zero is not an asymptotic value of the function $[u_1(z) - f(z)] : v_1(z)$, then the number of nontrivial zeros is equal to $2p$.

Theorem 3. *If at two neighboring points ξ the function $f(z)$ assumes values of different signs or, more generally, if at all points ξ at which $f(\xi) \neq 0$, the ratio $f(\xi) : u_1(\xi)$ has one and the same sign, then the function $f(z)$ has no more than $2p$ nontrivial zeros.*

Proof. Choose λ so that $0 \leq \lambda f(\xi) : u_1(\xi) < 1$; then the function $f_1(z) = u_1(z) - \lambda f(z)$ satisfies the conditions of Theorem 1 and $f(z) = \lambda^{-1}[u_1(z) - f_1(z)]$ has no more than $2p$ nontrivial zeros.

Remark. If zero is not an asymptotic value of the function $f(z) : v_1(z)$, then the function $f(z)$ has $2p$ nontrivial zeros.

Theorem 4. *Let $f(z)$ be a real entire function of exponential type not exceeding σ , $L_f < +\infty$ (see the footnote to Theorem 2), and let $\{\xi\}$ be a $(\sigma; 0)$ -regular sequence. If the function $f(z)$ assumes values of alternating signs on a set consisting of the points x_1, x_2 and the sequence $\{\xi\}$, then $f(z)$ is identically equal to zero.*

Proof. Remove two neighboring points from the set $x_1, x_2, \{\xi\}$. The remaining sequence will be $(\sigma; 0)$ -regular, and the number of nontrivial zeros of $f(z)$ with respect to this sequence is not less than 2, which is impossible.

This theorem is analogous to the fact that a polynomial of degree n cannot assume values of alternating signs at $n + 2$ points.

4°. In the theory of approximation of continuous functions by polynomials, a well-known theorem of Vallée-Poussin plays an important role. The preceding

theorems make it possible to obtain analogues of this theorem for entire functions. From Theorem 4 it follows:

Theorem 5. *Let $f(z)$ and $\varphi(z)$ be two real entire functions of exponential type not exceeding σ ; $L_f, L_\varphi < \infty$ (see the footnote to Theorem 2); let $\{\xi\}$ be a $(\sigma; 0)$ -regular sequence, and let $F(x)$ be an arbitrary real function. If for some x_1 and x_2 at the points of the sequence $\{x_1, x_2, \{\xi\}\}$ the difference $F(x) - \varphi(x)$ assumes values of alternating signs, then at least at one point of this sequence*

$$[F(x) - f(x)] : [F(x) - \varphi(x)] > 1. \quad (9)$$

From Theorem 3 it follows

Theorem 6. *Let $\{\xi\}$ be a $(\sigma; p)$ -regular sequence and let A be such a family of real entire functions of exponential type not exceeding σ , bounded on the real axis (see the footnote to Theorem 2), that the difference of any two functions from A has at least $2p + 1$ nontrivial zeros with respect to $\{\xi\}$. Let $F(x)$ be an arbitrary real function and $\varphi(z) \in A$. If the difference $F(x) - \varphi(x)$ has alternating signs at the points $\{\xi\}$, then for any function $f(z)$ from A at least at one point $\{\xi\}$*

$$[F(\xi) - f(\xi)] : [F(\xi) - \varphi(\xi)] > 1. \quad (9)$$

Example of a family A . The family A consists of all real entire functions of exponential type not exceeding σ satisfying the following conditions: 1) the values of the successive derivatives of orders $k_1, k_1 + 1, \dots, k_1 + 2l_1 - 1, \dots, k_\mu, k_\mu + 1, \dots, k_\mu + 2l_\mu - 1$ at the real points x_1, \dots, x_μ have prescribed values; 2) the functions

or their derivatives of certain orders take prescribed values at the $2j$ complex points $\xi_1, \bar{\xi}_1, \dots, \xi_j, \bar{\xi}_j$.

According to Theorem 6 of [1], the difference of two functions of this family has at least $2(j + l_1 + \dots + l_\mu)$ nontrivial zeros with respect to any sequence not containing the points x_1, x_2, \dots, x_μ . As the sequence $\{\xi\}$ one may take any such (σ, ρ) -regular sequence for which $p < j + l_1 + \dots + l_\mu$.

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Note: Figure translations are in progress. See original paper for figures.

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