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Abstract

Full Text

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ON THE PROBLEM OF INTERPOLATING FUNCTIONS OF n VARIABLES

To find an approximate representation of a function $\varphi(x)$ of n variables by elements from some finite set, one may use the values that this function assumes on some finite set of points

$$x^{(k)}, \qquad k = 1, 2, \dots, N. \tag{1}$$

The corresponding problem is called the interpolation problem, and the points $x^{(k)}$ are called the interpolation nodes. Most often one encounters interpolation by means of linear combinations of some set of functions:

$$\varphi(x) \cong \sum_{\nu=1}^{M} a_\nu \varphi_\nu(x). \tag{2}$$

Often, for $\varphi_\nu(x)$ one takes all possible monomials x^α of degree not exceeding m (here x^α denotes $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$). The number of such monomials is

$$M = (m+n)!/m!n!.$$

The values of any function $\varphi(x)$ at the points $x^{(k)}$ form an N-dimensional row vector

$$\varphi^k = \varphi(x^{(k)}), \qquad k = 1, 2, \dots, N. \tag{3}$$

Let us renumber all integer vectors $\alpha(\alpha_1, \alpha_2, \dots, \alpha_n)$ with nonnegative components such that $|\alpha| = \alpha_1 + \dots + \alpha_n \leq m$. The set of values of all monomials $x^{(k)\alpha_j}$ forms the matrix

$$S_{j,k} = (x^{(k)\alpha_j}) \tag{4}$$

with N columns and M rows. An arbitrary polynomial $Q = \sum_{j=1}^{M} a_j x^{\alpha_j}$ will be regarded as equivalent to the row vector $a(a_1, a_2, \dots, a_M)$. The values of this polynomial at the points $x^{(k)}$ will, evidently, constitute the vector

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$$Q^{(k)} = aS. \tag{5}$$

The interpolation problem consists in solving equation (5) for the vector a for a given $Q^{(k)}$, i.e., in reconstructing the polynomial Q from its values at the points $x^{(k)}$.

The solution of equation (5) and the polynomial Q are determined uniquely if $r(S) = M$, which is possible when $N \geq M$. In this case there exists at least one right inverse matrix S_d^{-1} for S. Then

$$a = aSS_d^{-1} = Q^{(k)}S_d^{-1}. \tag{6}$$

Each matrix S_d^{-1} will be called an interpolation matrix. For any polynomial Q

$$Q = Q^{(k)}S_d^{-1}x^{\alpha} \tag{7}$$

(where x^{α} is the vector $(x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_M})$). Formula (7) will be called the interpolation formula for the nodes $x^{(1)}, \dots, x^{(N)}$. If $N > M$, then there will be an infinite set of interpolation matrices and, consequently, of interpolation formulas. A prescribed vector $Q^{(k)}$ can serve as the vector of values of a polynomial in the case where

$$r\begin{pmatrix} S \\ Q^{(k)} \end{pmatrix} = r(S). \tag{8}$$

Condition (8) is the solvability condition for equation (5). The right-hand side of (7), however, has meaning for an arbitrary vector $Q^{(k)}$. Substituting into it the vector $\varphi^{(k)}$ in place of $Q^{(k)}$, we obtain the polynomial

$$P_{\varphi} = \varphi^{(k)}S_d^{-1}x^{\alpha} = \sum_{k=1}^{N} C_k(x)\,\varphi(x^{(k)}). \tag{9}$$

We shall call P_{φ} the interpolation polynomial for the function φ. It may happen that the solution of (5) exists for arbitrary $Q^{(k)}$; this means that $r(S) = N$ and, hence, $N \leq M$. In this case there exists at least one left inverse matrix S_g^{-1} for S

$$Q^{(k)}S_g^{-1}S = Q^{(k)}. \tag{10}$$

We shall not dwell on the cases $r(S) < N$, $r(S) < M$. For $r(S) = M = N$ the solution of (5) exists and is unique, and the interpolation problem is called classical. As an example of the solution of the classical interpolation problem one may point to the problem of interpolation by values at the nodes of a certain

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structure of parallelepipeds, which we shall call Newtonian. Let s be an integral n-dimensional vector with nonnegative components, and let $s-1$ be the vector $(s-1) = (s_1 - 1,\ s_2 - 1, \dots,\ s_n - 1)$. To each s such that $|s| \le m$ we put into one-to-one correspondence a parallelepiped Π_s, consisting of points of a certain cubic lattice with step h and edge lengths respectively $s_1 h,\ s_2 h, \dots,\ s_{nh}$. Let $s^{(1)} < s^{(2)}$ mean $s_k^{(1)} \le s_k^{(2)}$, with the inequality $s_k^{(1)} < s_k^{(2)}$ holding for at least one k.

We shall say that the parallelepipeds Π_s form a structure if:

D. From $s^{(1)} < s^{(2)}$ it follows that $\Pi_{s^{(1)}} \subset \Pi_{s^{(2)}}$.

Let the coordinates x_k of the parallelepiped Π_s be

$$x_k = \xi_{s_k} + lh; \qquad l = 0, 1, \dots, s_k.$$

(From property D it follows that ξ_{s_k} cannot depend on the other components of s.)

Construct a system of polynomials P_s by the formula

$$P_s(x) = \frac{\Gamma((x - \xi_{s-1})/h + 1)}{\Gamma((x - \xi_{s-1})/h - s + 1)} \equiv \prod_{k=1}^{n} \left\{ \prod_{l=0}^{s_k - 1} \left(\frac{x_s - \xi_{s_k - 1}}{h} - l \right) \right\} \tag{11}$$

(those k for which $s_k = 0$ do not enter the product). The polynomial $P_s(k)$ is the product of factorial polynomials in each variable. It vanishes at all points belonging to any parallelepiped Π_{s^*}, if at least one of the inequalities $s_i^* < s_k$ is satisfied.

Next, let the operator

$$\Delta_s = \Delta_1^{s_1} \Delta_2^{s_2} \cdots \Delta_n^{s_n} \tag{12}$$

is the product of difference operators of orders respectively s_k in all variables x_k, computed at the points:

$$x_k = \xi_{s_k - 1} + lh, \quad h = 0, 1, \dots, s_k - 1. \tag{13}$$

It is easily verified that

$$\Delta_s P_q = 0, \quad s \ne q; \qquad \Delta_s P_q = s!, \quad s = q. \tag{14}$$

Any polynomial of degree m is uniquely representable in the form

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$$Q = \sum_{j=1}^{M} a_{s_j} P_{s_j};$$

its coefficients a_s can, by virtue of (14), be expressed in the form

$$a_s = \Delta_s Q / s!, \tag{15}$$

i.e., found from the values of Q at all nodes of the structure. The interpolation problem in this case is classical, since the number of nodes is exactly M and $r(S) = M$.

For an arbitrary function φ we have the interpolating polynomial, constructed from its values at the points of the structure,

$$P_\varphi = \sum_{|s| \le m} \frac{\Delta_s \varphi}{s!} P_s(x). \tag{16}$$

An important problem in the theory of interpolation is to find the maximum error of the interpolation formula $\varphi(x) = P_\varphi(x)$ in a certain class of functions. The value of this error at a certain point z is a functional on the function φ:

$$(j, \varphi) \equiv \varphi(z) - P_\varphi(z) = \varphi(z) - \sum_{k=1}^{N} C_k(z) \varphi\left(x^{(k)}\right), \tag{17}$$

where $C_k(z) = S_d^{-1} z^\alpha$.

The quantities $C_k(z)$, obviously, are connected by the linear conditions

$$(j, x^{\alpha_s}) = 0, \quad s = 1, 2, \dots, M. \tag{18}$$

The functional (j, φ) is a bounded linear functional in the space $W_2^{(m)}$ of functions whose derivatives of order m are square-integrable. Therefore it is convenient to consider the maximum of this functional on the unit sphere in this space.

Problem 1. Find

$$\max_{\|\varphi\|_{W_2^{(m)}} = 1} (j, \varphi). \tag{19}$$

This is solved in the same way as the problem of finding the maximum error of a cubature formula. Repeating the arguments given in the paper ([1]), we find that the extremal function is a solution of the equation

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$$\Delta^m u = (-1)^m \left[\delta(x-z) - \sum_{k=1}^{N} C_k(z) \delta\big(x - x^{(k)}\big) \right], \tag{20}$$

satisfying the conditions

$$B^{(t)} u\big|_{\Gamma} = 0, \quad t = 1, 2, \dots, m. \tag{21}$$

The extremal function, and with it the greatest possible error, thus depend on the domain in which the class of functions under consideration is specified. We have the general estimate:

$$|(j, \varphi)| \leq K(\Omega) \|\varphi\|_{W_2^{(m)}(\Omega)}. \tag{22}$$

Let Ω_2 be narrower than Ω_1: $\Omega_2 \subset \Omega_1$. Then the extremal function in Ω_1 is also defined in Ω_2, satisfies there the same equation (20), but, possibly, will no longer be extremal there. Moreover, its norm in this domain will be less than one. Thus,

$$\max_{\|\varphi\|_{W_2^{(m)}(\Omega_1)}=1} (j, \varphi) \leq \max_{\|\varphi\|_{W_2^{(m)}(\Omega_2)}=1} (j, \varphi). \tag{23}$$

Consequently, the constant $K(\Omega)$ decreases as the domain is enlarged. Let us show that it tends to a certain definite limit K^{∞} under unbounded enlargement of the domain. For this it is enough to establish that equation (20) has a solution in the whole space belonging to $W_2^{(m)}$.

Let $G(x)$ be a fundamental solution of the equation

$$\Delta^m G = (-1)^m \delta(x). \tag{24}$$

As is known,

$$G = \varkappa r^{2m-n}, \quad n \text{ odd}; \qquad G = \varkappa r^{2m-n} \lg r / 2\pi i, \quad n \text{ even}. \tag{25}$$

We shall show that the function

$$\psi(x) = G(x-z) - \sum_{k=1}^{N} C_k(z) G(x - x^{(k)}) \tag{26}$$

belongs to $W_2^{(m)}$ and, consequently, is the unique extremal element in $W_2^{(m)}$ for the functional (j, φ).

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For this purpose we expand the derivatives of order m of the function $G(x^{(0)}-x)$ in a power series in x in some ball of radius $A > \max[|z|, |x^{(k)}|]$. We have:

$$D^\beta G(x^{(0)}-x) = \sum_{|\alpha|\le m} \frac{D^{\beta+\alpha}G}{\alpha!}\bigg|_{x-x^{(0)}} x^\alpha + R(x, x^{(0)}) = Q^{(\beta)}(x, x^{(0)}) + R(x, x^{(0)}), \tag{27}$$

where $Q^{(\beta)}$ is a polynomial of degree m in x, and $R^{(\beta)}(x, x^{(0)})$ satisfies the inequality

$$R^{(\beta)}(x, x^{(0)}) \le K r_0^{-n} \lg r_0, \tag{28}$$

valid for sufficiently large r_0. In view of condition (18) we have $(j, D^\beta G(x^{(0)}-x)) = (j, R^{(\beta)}(x, x^{(0)}))$, whence

$$\left|(j, D^\beta G(x^{(0)}-x))\right| = \left|D^\beta \psi(x^{(0)})\right| \le \left[1 + \sum |C_k|\right] K r_0^{-n} \lg r_0. \tag{29}$$

From the inequality obtained it follows that the integral $\int |D^\beta \psi|^p \, dx$ converges for any p greater than one, and hence that the norm of $\psi(x)$ is bounded. Of interest is the problem of finding an optimal interpolation formula for given nodes $x^{(k)}$, if $N > M$. In this case one may use the arbitrariness in the definition of S_d^{-1} to minimize the error.

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Note: Figure translations are in progress. See original paper for figures.

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