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# MATHEMATICS

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1961

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**Abstract**

**Full Text**

MATHEMATICS

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## THE SET OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION

*(Presented by Academician V. I. Smirnov, 20 X 1960)*

1°. The present note is devoted to the description of all (continuous and of bounded variation) solutions of the following equations:

$$M'(t) = A(t)M(t) \quad \text{a.e.} \quad (1)$$

with the initial condition

$$M(a) = M_0 \quad (2)$$

and

$$x'(t) = A(t)x(t) \quad \text{a.e.} \quad (3)$$

with the initial condition

$$x(a) = x_0. \quad (4)$$

Here  $t$  is a real argument ranging over the finite interval  $[a, b]$ ; the values  $x(t)$  belong to the Banach space  $X$ ; the values  $M(t)$  and  $A(t)$  belong to a normed ring  $U$  with identity. In considering equation (3) we put  $U = \{X \rightarrow X\}$ , the ring of bounded linear operators from  $X$  into  $X$ .

If  $Y$  is some Banach space, then  $CV[Y]$  will denote the class of all continuous functions of bounded variation on  $[a, b]$  with values in  $Y$ ;  $S[Y]$  is the class of all singular functions on  $[a, b]$  that vanish for  $x = a$ .\*

2°. Let

$$G_0(t) = \int_a^t A(\tau) d\tau$$

and  $G(t) = G_0(t) + H(t)$ , where  $H \in S[U]$ . Form the multiplicative Stieltjes integral

$$M(t) = \prod_a^t (E + dG) = \lim \prod_{i=0}^{n-1} (E + \Delta G(t_i)), \quad (5)$$

where a Riemann-type limit is meant, and the factors in the product  $\prod_{i=0}^{n-1}$  are written in the order of increasing indices from right to left. This integral always exists for a function  $G \in CV[U]$ .  $G(t)$  is uniquely recovered from  $M(t)$  by the formula

$$G(t) = \int_a^t dM M^{-1}$$

(see (2)).

The functions  $G(t)$  and  $M(t)$  are absolutely continuous or singular only simultaneously, and have a derivative for a given value of  $t$  only simultaneously.

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\* Continuity is understood in the sense of the topology generated by the norm. Bounded variation of a function  $y(t)$  means boundedness of the sums  $\sum \|y(t_{i+1}) - y(t_i)\|$ . A function  $y(t)$  is singular if it belongs to  $CV[Y]$ ,  $y'(t) = 0$  a.e., and  $y(t) \neq \text{const}$ . The derivatives appearing in (1) and (3) are understood in the sense of convergence in norm.

$M(t)$  satisfies equation (1) and the initial condition

$$M(a) = E. \quad (6)$$

Any solution of problem (1), (6) belonging to  $CV[U]$  has the form (5), and, consequently, the general solution of this problem depends on an arbitrary normalized singular function  $H(t)$  with values in  $U$ .

For the general (in the class  $CV[U]$ ) solution  $\bar{M}(t)$  of equation (1) under the zero initial condition, one can give two representations:

$$\bar{M}(t) = \prod_a^t (E + dG) - \prod_a^t (E + dG_0), \quad (7)$$

$$\bar{M}(t) = \int_a^t \prod_s^t (E + dG_0) dL(s) = L(t) + \int_a^t \prod_s^t (E + dG_0) A(s) L(s) ds, \quad (8)$$

where  $L(s)$  is an arbitrary function of class  $S[U]$ .

**Remark.** Since the classes of functions representable by formulas (7) and (8) coincide, we obtain a one-to-one mapping  $H \sim L$  of the class  $S[U]$  onto itself. This mapping is defined by the integral equation

$$L(t) + \int_a^t \prod_s^t (E + dG_0) A(s) L(s) ds = \prod_a^t (E + dG) - \prod_a^t (E + dG_0). \quad (9)$$

Equation (9) is easily solved:  $L(t)$  is the singular component of the right-hand side. Taking into account that each of the two terms on the right-hand side satisfies equation (1), we obtain

$$L(t) = \left[ \prod_a^t (E + dG) - \prod_a^t (E + dG_0) \right] - \int_a^t A(\tau) \left[ \prod_a^\tau (E + dG) - \prod_a^\tau (E + dG_0) \right] d\tau. \quad (10)$$

The general solution (in the class  $CV[U]$ ) of problem (1), (2) evidently has the representation

$$M(t) = \prod_a^t (E + dG_0) M_0 + \left[ \prod_a^t (E + dG) - \prod_a^t (E + dG_0) \right]. \quad (11)$$

The question arises whether the expression

$$M(t) = \prod_a^t (E + dG) M_0 \quad (12)$$

will also provide the general solution.

With the aid of relation (9) it is shown that this will be so if and only if the element  $M_0$  has a left inverse.

Thus, two theorems are valid:

**Theorem 1.** The general solution in the class  $CV[U]$  of problem (1), (2) is (11), or, in another form,

$$M(t) = \prod_a^t (E + dG_0) M_0 + L(t) + \int_a^t \prod_s^t (E + dG_0) A(s) L(s) ds. \quad (11')$$

Here  $G = G_0 + H$ , and the functions  $H(t)$  and  $L(t)$  are arbitrary functions of class  $S[U]$ , the relation between which is established by formulas (9) or (10).

**Theorem 2.** In order that formula (12) give the general (in the class  $CV[U]$ ) solution of problem (1), (2), it is necessary and sufficient that the element  $M_0$  have a left inverse.

**Remark.** From Theorem 3 given below it follows that, for  $M_0 \neq 0$ , every solution (in the class  $CV[U]$ ) of problem (1), (2) can be represented in the form (12), but with a function  $G(t)$  whose values lie in the ring  $\{U \rightarrow U\}$  containing  $U$ .

3°. Let us pass to problem (3), (4), while preserving the notation of item 2°, taking for  $U$  the ring  $\{X \rightarrow X\}$ .

The general solution in the class  $CV[X]$  of problem (3), (4), as is easy to verify, is

$$x(t) = \prod_a^t (E + dG_0)x_0 + \xi(t) + \int_a^t \prod_s^t (E + dG_0)A(s)\xi(s) ds, \quad (13)$$

where  $\xi(s)$  is an arbitrary function of class  $S[X]$ . If one chooses and fixes  $\bar{x} \in X$ ,  $\bar{x} \neq 0$ , then a solution of problem (3), (4) will also be the function

$$x(t) = \prod_a^t (E + dG_0)x_0 + \left[ L(t) + \int_a^t \prod_s^t (E + dG_0)A(s)L(s) ds \right] \bar{x}, \quad (14)$$

where  $L(t)$  is, as before, an arbitrary function of class  $S[U]$ . It turns out that (14) is the general (in the class  $CV[X]$ ) solution of problem (3), (4).

(14) can be rewritten in the form

$$x(t) = \prod_a^t (E + dG_0)x_0 + \left[ \prod_a^t (E + dG) - \prod_a^t (E + dG_0) \right] \bar{x}, \quad (15)$$

where  $G = G_0 + H$  and  $H \sim L$ .

If  $x_0 \neq 0$ , then as  $\bar{x}$  one may take  $x_0$ . Consequently, the general solution in the class  $CV[X]$  of problem (3), (4) for  $x_0 \neq 0$  is

$$x(t) = \prod_a^t (E + dG)x_0. \quad (16)$$

Thus, the following is true:

**Theorem 3.** *The general solution in the class  $CV[X]$  of problem (3), (4) is (15), or, in another form, (13). If  $x_0 \neq 0$ , then the general solution can also be written in the form (16). Here  $\bar{x}$  is a fixed nonzero element of  $X$ ,  $\xi(s)$  is an*

arbitrary function of class  $S[X]$ ,  $G = G_0 + H$ , and  $H(t)$  is an arbitrary function of class  $S[U]$ .

4°. Let us now consider the equation “in differentials”

$$dM(t) = dG(t)M(t), \quad (17)$$

to which it is natural to assign the following meaning:

$$\|M(t+h) - M(t) - [G(t+h) - G(t)]M(t)\| = o\left(\bigvee_t^{t+h} G\right).$$

Equation (17) with the initial condition (6) is equivalent to the integral equation

$$M(t) = E + \int_a^t dG M. \quad (18)$$

Equation (18) was considered by Wall <sup>(1)</sup> (for the finite-dimensional case) and Mac Nerney <sup>(2)</sup>. They showed that the solution of (18) exists, is unique, and is representable in the form (5).

Let us return to problem (1), (6). Associate with the function  $A(t)$  the family of its normalized primitives  $G(t) = G_0(t) + H(t)$ , where  $H$  ranges over  $S[U]$ . Having fixed the choice of a primitive  $G(t)$  and substituted it into (17), we obtain an equation in differentials whose solution will also be a solution of (1). In other words, equation (1) “stratifies” into a family of equations (17). By choosing the primitive  $G(t)$  for substitution into equation (17), we thereby choose one of the solutions of equation (1).

The case of the problem with an arbitrary initial condition presents no special features here. The general solution of the problem (17), (2) is (12). Hence it follows that if (17) is generated by equation (1) and if  $M_0$  has no left inverse, then problem (1), (2) has a solution which, for no choice of an antiderivative for  $A(t)$  (in the class  $CV[U]$ ), will be a solution of problem (17), (2) (cf. the remark to Theorem 2).

**Remark.** Equation (17) may fail to be associated with equation (1). This will be the case if the continuous function of bounded variation  $G(t)$  has no derivative almost everywhere.

In an analogous way one considers the equation

$$dx(t) = dG(t)x(t).$$

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Received  
19 IX 1960

## References

<sup>1</sup> H. S. Wall, *Arch. Math.*, **5**, 1–3, 160 (1954). <sup>2</sup> J. S. MacNerney, *Ann. Math.*, **61**, 2, 354 (1954).

*Note: Figure translations are in progress. See original paper for figures.*

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